Math 615, Winter 2017 **Problem Set #5 Solutions**

1. A separable algebra is the directed union of its finitely generated separable subalgebras, and a direct limit of reduced rings is reduced. Hence, we may assume that each of the separable algebras is finitely generated over K, and then embeds in a finite product of separable domains. Hence, we may also assume that both of the rings are domains. A finitely generated domain over K is separable if and only if its localization at one nonzero element is smooth. Hence, both can be enlarged by localization to be smooth, and then the tensor product is smooth over K and so reduced. There are a great many solutions. $\square$

2. $\frac{\partial F}{\partial x} = xF_1$, where $F_1 = 2y^2 + nx^{n-2}$ and $\frac{\partial F}{\partial y} = yF_2$, where $F_2 = 2x^2 + ny^{n-2}$. Since x, F_1 have no common factor with y or F_2, xF_1, yF_2 is a regular sequence and so a system of parameters in $\mathbb{C}[[x,y]]$, and $\mathbb{C}[[x,y]]/I$, where $I = (\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y})$, is Artin.
$F - \frac{1}{n}(x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y}) = \frac{2}{n}x^2y^2$. If $F \in I$ then $x^2y^2 \in I$ as well, and this gives $x^2y^2 = GxF_1 + HyF_2$. Since x does not divide F_2, we have $H = xH_1$ and, similarly, $G = yG_1$. Factoring out xy yields $xy = (2y^2 + nx^{n-2})G_1 + (2x^2 + ny^{n-2})H_1$, which is clearly false (compare degree two terms on both sides). Thus, $F \notin I$. Similarly, to see that $(*)$ $xF \in I$, it suffices to see that $x(x^2y^2) \in I$, and, hence, to see that $x^2y \in (F_1, F_2)$. Mod F_2, x^2y is a scalar times y^{n-1}, while $nx^{n-2} \equiv y^2v$ with $v \in (x,y)$, so that $2y^2 + nx^{n-2} \equiv uy^2$ where $u = 2 + v$ is a unit of $\mathbb{C}[[x,y]]$, and $(*)$ follows. By symmetry, $yF \in I$. Thus, $\overline{F} \neq 0$ while $\overline{F}^2 = 0$ in $\mathbb{C}[[x,y]]/I$. Evidently, $d\overline{F} = 0$ in R/I, since $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ are 0 in $\mathbb{C}[[x,y]]/I$. $\square$

3. (a) I^2 is spanned as an abelian group by products fg with $f, g \in I$, and $d(fg) = gdf + fdg \in I\Omega_{T/R}$, so that I^2 is killed. If $g \in I$ and $f \in T$, $d(fg) = fdg + gdf$, and the second term is 0 in $\Omega_{T/R}/I\Omega_{T/R}$. This implies S-linearity.

(b) To give the R-algebra splitting $T/I \to T/I^2$ of the map $T/I^2 \to T/I$, for each element of $t \in T$ one must specify $\phi([t]) \in T/I^2$, and we also write $\phi(t)$ for $\phi([t])$. The element $\phi(t)$ must have the form $\overline{t} - \delta t$ where $\delta(t) \in I/I^2$. In order for the function ϕ corresponding to a map δ to preserve addition, δ must be linear. In order for ϕ to preserve multiplication, we must have $(\overline{t} - \delta(t))(\overline{t'} - \delta(t')) = \overline{tt'} - \delta(tt')$, and this is simply the condition that $\delta(tt') = t\delta(t') + t'\delta(t)$. Thus, δ is a derivation of T into I/I^2. In order that ϕ be well-defined on T/I, we must have that for $f \in I$, that $(\dagger)$ $\delta(f) = \overline{f}$, so that $\phi(f) = 0$. In order for ϕ to be R-linear, we must have that δ be an R-derivation of T into I/I^2. Thus, giving a splitting ϕ is equivalent to giving an R-derivation $\delta : T \to I/I^2$ such that $(\dagger)$ $\delta f = \overline{f}$ for all $f \in I$. Such an R-derivation corresponds to a T-linear map $\widetilde{\theta} : \Omega_{T/R} \to I/I^2$ such that $(\dagger')$ $\theta(df) = \overline{f}$ for all $f \in I$, and, hence, to giving a T-linear map $\theta : \Omega_{T/R}/I\Omega_{T/R} \to I/I^2$ such that $(\dagger'')$ $\theta([df]) = \overline{f}$ for all $f \in I$. The T-linearity of θ is equivalent to its S-linearity, since I kills both modules. But this is exactly the condition that θ be a splitting over S of the map described in part (a).

(c) is immediate, since, quite generally, $A \to B \to C \to 0$ is split exact if and only if $A \to B$ is injective and the surjection $B \twoheadrightarrow C$ splits. $\square$

4. The specified elements are not even in the expansion of m (which is no longer maximal), since this is the union of the expansions of the ideals $(x_1, \ldots, x_h)$, as h runs through the positive integers, and the image of each of these elements is nonzero modulo every ideal of the form $(x_1, \ldots, x_h)$.

5. Map $\Lambda_0[Y_1.Y_2] \twoheadrightarrow \Lambda_0[t^a \cos t,\, t^b \sin t]$. Since $\cos t$ is transcendental over $\mathbb{R}[t]$ (it has infinitely many roots), and $\cos^2 t + \sin^2 t = 1$, $\dim(S) = 2$, and the kernel must be a height one, hence, principal prime. Let $M = \max\{a, b\}$. Then the generator of the kernel is $F = t^{2M-2a}Y_1^2 + t^{2M-2b}Y_2^2 - t^{2M}$, since it is in the kernel and is irreducible (the coefficient of Y_1 or Y_2 is one, and the sum other terms is not the negative of a perfect square). The Jacobian $\begin{pmatrix}\frac{\partial F}{\partial Y_1} & \frac{\partial F}{\partial Y_2}\end{pmatrix} = \begin{pmatrix}t^{2M-2a}2Y_1 & t^{2M-2b}2Y_2\end{pmatrix}$ evaluated at $Y_1 = t^a \cos t$, $Y_2 = t^b \sin t$ gives $\begin{pmatrix}t^{2M-2a}2t^a \cos t & t^{2M-2b}2t^b \sin t\end{pmatrix}$. The orders of the two entries in $\mathbb{R}[[t]]$ are $2M - 2a + a$ and $2M - 2b + b + 1$: we want the minimum of these two. If $a \geq b$, these are a and $2a - b + 1 \geq a + 1$, and the minimum is $\ell_S = a$. If $a < b$, these are $2b - a \geq b + 1$ and $b + 1$, and the minimum is $\ell_S = b + 1$. The generators of the prime ideal $Q = t\Lambda \cap S$ are $t, t^a \cos t, t^b \sin t$ if $a \geq 1$ (even if $b = 0$, since $\frac{\sin t}{t} \in \mathbb{R}[[t]]$), while if $a = 0$ one gets $t, 1 - \cos t, t^b \sin t$. The elements that generate S' over S are obtained by dividing these by t (of course, one does not need to adjoin $\frac{t}{t} = 1$).

6. The module C_n obtained from $De_1 \oplus \cdots \oplus De_{n+1|}$ by killing the relations $e_j - p_j e_{j+1}$, $1 \leq j \leq n$, is free on the image of e_{n+1}. The other e_j are identified with multiples of e_{n+1}: e_j is identified with $p_j \cdots p_n e_{n+1}$, and so C_n may be identified with rank one free D-submodule of L generated by $f_n = 1/(p_1 \cdots p_n)$. Here, e_1 is mapped to $f_0 = 1$, e_2 to $f_1 = 1/p_1$, e_3 to $f_2 = 1/(p_1p_2)$, $\ldots$, e_j to f_{j-1}, $\ldots$, e_{n+1} to f_n. Taking a direct limit shows that C may be identified with the submodule of L generated by all the f_n, which is clearly the same as the set of all fractions that may be written with a square-free denominator. If we localize at $P = P_i$, so that all p_j for $j \neq i$ become invertible, this simply becomes the free rank one D_P module on the generator $1/p_i$, and so the map splits. It cannot split over D, because if it does, the image of $1 \in C$ in the free module will contain a nonzero element of a free D-module that is divisible by every p_i. Such a nonzero element has a nonzero coordinate, which would be a nonzero element of D divisible by every p_i. But a nonzero element of D can only be divisible by finitely many p_i, a contradiction. $\square$

EC 9. If we localize at a prime of D, say P, all but at most one of the primes p_i becomes invertible in D_P. That the images of X_j and X_{j+1} are associates (each is a unit times the other) unless $j = i$ and p_i is not invertible. In that case, it is still true that the image of x_i is p_i times the image of x_{i+1}, and so S_P may be identifieid with $D_P[x_{i+1}]$, and is smooth over D_P and formally smooth over D. Hence, every S_Q is formally smooth over D, since Q lies over some prime P of D. On the other hand, one may deduce that S is not formally smooth over D from problems 3. and 6. $(T/I) \otimes \Omega_{T/D}$ is the free S-module on the images of the dX_i. Since $d(X_i - p_iX_{i+1}) = dX_i - p_i dX_{i+1}$, the map $\psi : \Omega_{T/D}/I\Omega_{T/D} \twoheadrightarrow \Omega_{S/D}$ that occurs on the right in the sequence $(*)$ in Problem 3. may be described as $\mathrm{id}_S \otimes_D \zeta$, where $\zeta : G \to C$ is the map studied in Problem 6. There is a D-algebra retraction $S \twoheadrightarrow D$ that kills the images of all of the X_j. If ψ were split, we could use this S-algebra structure on D to apply $D \otimes_S \_$, and it would follow that ζ is split over D, a contradiction. $\square$

EC 10. (a) The tensor product is a finite dimensional $\mathbb{C}$-algebra, and is clearly local, since the elements of the ring not in the tensor product of the t copies of $\mathbb{C}$ are spanned by elements $u_1 \otimes \cdots \otimes u_h$ where are least one u_ν is nilpotent. When we expand g_t^t using the distributive law, there is only one term which occurs without exponent at least two: the terms where the exponent is 2 or more evidently vanish. That one term, $f_1 f_2 \cdots f_t$, is not zero: it corresponds to the tensor product of $f_1 \otimes f_2 \otimes \cdots \otimes f_t$, and these are all nonzero in their respective factors. On the other hand, in the expansion of g_t^{t+1} every term has a factor which occurs with exponent two or more, and each $f_\nu^2 = 0$.

(b) Note that $\mathrm{Hom}_{\mathbb{C}}(A_t, \mathbb{C})$ is spanned over $\mathbb{C}$ by the maps L_i, $0 \le i \le t-1$, where $L_i(t^i) = 1$ and $L_i(t^j) = 0$ if $0 \le j \le t-1$ and $j \neq i$. Note that $t^{t-1-i} L_{t-1} = L_i$, $0 \le i \le t-1$. Hence, the $\mathrm{Hom}_{\mathbb{C}}(A_t, \mathbb{C})$ is cyclic of the same dimension over $\mathbb{C}$ at A_t, and so must be $\cong A_t$. The remaining statements follow as indicated.

(c) We may consider R as an A_t algebra since we have $A_t \cong \mathbb{C}[u] \subseteq R$ with z mapping to u. We may consider B_t as an A_t-algebra by sending $z \mapsto g_t$. Then the surjection $R \otimes_{\mathbb{C}} B_t \to R \otimes_{A_t} B_t$ kills $u - g_t$, since u and g_t are the images of z, and z passes through the tensor symbol. This yields a surjection $R \otimes_{\mathbb{C}} B_t/(u - g_t) \to R \otimes_{A_t} B_t$, and so it suffices to show that R injects into this tensor product. However, as an A_t module, we have that A_t splits from B_t, say $B_t \cong A_t \oplus N_t$, where N_t is an A_t -submodule of B_t, Then $R \otimes_{A_t} B_t \cong R \otimes_{A_t} (A_t \oplus N_t) \cong (R \otimes_{A_t} A_t) \oplus (R \otimes_{A_t} N_t) \cong R \oplus R \otimes_{A_t} N_t$, and so it is clear that R injects. The last statement is clear, since it holds even for g_t in B_t and u is identified with g_t.

(d) Choose a basis for the maximal ideal. By finitely many successive applications of part (c), the ring can be enlarged to an Artin local ring over $\mathbb{C}$ in which the differential of every element of the basis and, hence, of the original maximal ideal is 0. Iterate countably many times, to obtain an ascending sequence of Artin local rings over $\mathbb{C}$ such that the differential of every element of the maximal ideal of a given ring in the sequence is 0 when it is considered as an element of the next ring in the sequence. The direct limit of this sequence evidently has the required properties.