Math 615, Winter 2019 **Problem Set #1**
Due: Wednesday, February 6

1. Let $R \subseteq S$ be rings and W a multiplicative system in R. Let T be the integral closure of R in S. Show that the integral closure of $W^{-1}R$ in $W^{-1}S$ is $W^{-1}T$. Also show that if I is an ideal of R, the integral closure of $IW^{-1}R$ in $W^{-1}R$ is $\overline{I}W^{-1}R$, where $\overline{I}$ is the integral closure of I in R.

2. Let $n \geq 1$ be an integer, and let $R \subseteq S$ be an inclusion of rings that is compatible with $\mathbb{Z}^n$-gradings (or $\mathbb{N}^n$-gradings) on R, S. Prove that the integral closure of R in S is also compatible with these gradings. [Suggestion: this can be deduced from the result proved in class for the case $n = 1$.]

3. Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K. If $\alpha = (a_1, \ldots, a_n) \in \mathbb{N}^n$, let $x^\alpha = x_1^{a_1} \cdots x_n^{a_n}$, and let $L(x^\alpha) = \alpha$. If I is an ideal generated by monomials, let $L(I) = \{\alpha \in \mathbb{N}^n : x^\alpha \in I\}$. Note that there is a bijection between monomial ideals in R (the ideals generated by monomials or, equivalently, spanned over K by monomials) and subsets S of $\mathbb{N}^n$ such that if $\alpha \in S$ and $\beta \in \mathbb{N}^n$ then $\alpha + \beta \in S$. Show that I is integrally closed if and only if $L(I)$ contains every point of $\mathbb{N}^n$ in the convex hull of $L(I)$ over the rational numbers $\mathbb{Q}$.

4. If I is an ideal integrally closed in a ring R, prove that $IR[x]$ and $IR[x] + xR[x]$ are integrally closed in $R[x]$.

5. In the polynomial ring $R = K[x, y, z]$ in three variables over the field K, let $m = (x, y, z)R$, let $J = (x^4,\, y^5z^2)R$, and let I be the integral closure of J. Prove that x^3y^2z is integral over mI but not in mI. Thus, the product of an integrally closed monomial ideal and the maximal ideal in a polynomial ring need not be integrally closed.

6. Let R be a domain with fraction field F. Define an element $f \in F$ to be *quasi-integral* over R is there is a finitely generated R-submodule M of F that contains $R[f]$, i.e., such that M contains all powers of f. (If R is Noetherian, f is quasi-integral over R if and only if f is integral over R.) Prove that the set of elements of F quasi-integral over R is an R-algebra.

EXTRA CREDIT 1. Let R be a domain that contains a field K and let $a, b \in R - \{0\}$ be such that b is not a unit but $a \in b^sR$ for every positive integer s. Let L be the fraction field of R. Let t be a formal power series indeterminate over R. Show that $R[[t]]$ is not integrally closed. You may asusme that the characteristic of K is 0, but this is not needed. [Suggestion: pick an integer $n \geq 2$ and show that there is a power series $f \in L[[t]]$ corresponding to $b(1 + b^{-2}t)^{1/n}$. Show that $af \in R[[t]]$, that f is integral over $R[[t]]$, but that $f \notin R[[t]]$. If the characteristic p of K is positive, choose n such that p does not divide n.] This example is due to A. Seidenberg.

EXTRA CREDIT 2. Let D be a ring such that the localization of D at any prime ideal is either a Noetherian discrete valuation domain or a field. In particular, D has Krull dimension at most 1. Assume also that $\mathrm{Spec}\,(D)$ is connected. Must D be an integral domain? Prove your answer. (Note: it is *not* assumed that D is Noetherian.)