Math 615, Fall 2019 **Problem Set #1: Solutions**

1. For the first part, note that if s/w is integral over $W^{-1}S$, one may clear denominators in an equation of integral dependence and multiply by a further element of W to get an equation over R of the form $w_0s^n + r_1s^{n-1} + \cdots + r_0 = 0$ (the extra multiplication may be needed to make sure that we get 0, not just an element that is equivalent to 0). Multiply by w_0^{n-1} to get $(w_0s)^n + w_0r_1(w_0s)^{n-1} + \cdots + w_0^n r_0 = 0$, which shows $w_0s \in T$ and so $s/w = w_0s/(w_0w) \in W^{-1}T$. $\square$
For the second part, we may use the first part to see that the integral closure of $W^{-1}R[It]$ in $W^{-1}R[t]$ is $W^{-1}S$, where S is the integral closure of $R[It]$ in $R[t]$. Comparing the degree one parts (with respect to degree in t), shows that the integral closure of $IW^{-1}R$ is $\overline{I}W^{-1}R$. $\square$

2. We use induction on n. The case where $n = 1$ was done in class. If $n > 1$ and we know the result for $n-1$, first note that there is a compatible $\mathbb{Z}^{n-1}$ (or $\mathbb{N}^{n-1}$) -grading on both rings in which then $(h_1, \ldots, h_{n-1})$ component is the sum of all the $(h_1, \ldots, h_{n-1}, s)$-components as s varies. By the induction hypothesis, each element of the integral closure is a sum of elements u in which the first $n-1$ indices are constant. Now put on $\mathbb{Z}$ (or $\mathbb{N}$) -grading where the degree h component is the sum of all the $(s_1, \ldots, s_{n-1}, h)$ components as the first $n-1$ indices vary. The result now follows at once by applying the result for this grading and $n = 1$ to the elements u.

3. R is graded by $\mathbb{N}^n$ such that $\deg(x^\alpha) = \alpha$, and then $R_\alpha = Kx^\alpha$. The integral closure of an ideal is also therefore $\mathbb{N}$-graded by Problem **2.**, and it suffices to determine when x^β is integral over I. Consider an equation $F(z)$ of integral dependence of x^β on I of degree n. We get another such equation by taking only terms of degree $n\beta$. By dividing by a power of z, we may assume F has constant $\neq 0$. Let $\deg_z F = n$. Since the nonzero constant term in I^n has the same degree as $x^{n\beta}$, $x^{n\beta}$ is in I^n, and so $x^\beta \in \overline{I}$ iff $x^{n\beta} \in I^n$ for some n iff for some n, h monomials $x^{\delta_1}, \ldots, x^{\delta_h} \in I$, and $a_1, \ldots, a_h \in \mathbb{N}$ such that $a_1 + \cdots a_h = n$, we have $x^{n\beta} = (x^{\delta_1})^{a_1} \cdots (x^{\delta_h})^{a_h}$. The corresponding condition on exponent vectors is $n\beta = \sum_{i=1}^h a_h\delta_i$ for $\delta_i \in L(I)$, and dividing by n, we see that this is precisely equivalent to the statement that β is in the convex hull of $L(I)$ over $\mathbb{Q}$. $\square$

4. The integral closure of $IR[x]$ is homogeneous, and so it generated by elements of the form rx^n, $r \in R$. It suffices to show that for such an element in $\overline{IR[x]}$, one has $r \in I$. This follows by applying the R-algebra homomorphism $R[x] \to R$ that sends $x \mapsto 1$, which takes $IR[x]$ onto I and so maps the integral closure of $IR[x]$ into the integral closure of I, which is I. Similarly, for the elements of the integral closure of $IR[x] + xR[x]$, it suffices to see that an element $f \in R$ integral over this ideal is in I. In this case, one map apply R-homomorphism $R[x] \to R$ such that $x \mapsto 0$. $\square$

5. Note that mI contains mJ, which contains x^5, x^4y, and y^5z^3, and have that $(x^3y^2z)^3 = (x^5)(x^4y)(y^5z^3) \in (mI)^3$, so that $f = x^3y^2z$ is integral over mI. It suffices to show that x^3y^2z is not in mI, and for this it suffices to show that the elements $f/x = \lambda = x^2y^2z$, $f/y = \mu = x^3yz$ and $f/z = \nu = x^3y^2$ are not integral of (x^4, y^5z^2). λ and μ are not in the integral closure even if we specialize $z \mapsto 1$, $x \mapsto t^5$, $y \mapsto t^4$ in $K[t]$ and ν is not in even if we specialize $y \mapsto 1$ and $z \mapsto 0$. $\square$

6. Suppose that f, g are quasi-integral over R and $M = Ru_1 + \cdots + Ru_s$ is a finitely generated submodule of F containing all powers f and $N = Rv_1 + \cdots + Rv_t$ is a finitely generated R-submodule of F containing all powers of g. Then the R-submodule W of F generated by the st elements $u_i v_j$ is finitely generated and contains all products of elements of M with elements of N. Hence, it is closed under addition and contains all $f^i g^j$. Thus, it contains all powers of fg and also all powers of $f \pm g$. It follows that fg and $f \pm g$ are also quasi-integral over R. $\square$

EC1. To get the required power series root it suffices to show $(1+z)^{1/n}$ has a power series expansion with coefficients in R if p does not divide n. We may then substitute $b^{-2}t$ for z. Newton's binomial theorem provides such a root where the coefficient of z^k is $a_k/k!$ where $a_k = (\frac{1}{n})(\frac{1}{n} - 1)\cdots(\frac{1}{n} - k + 1)$. There is no problem if R contains the rationals. If R contains $\mathbb{Z}/p\mathbb{Z}$ and p does not divide n, it suffices to see that when a_k is written in lowest terms the denominator is not divisible by p, i.e., that the p-adic order of the numerator is at least that for the denominator. Choose an integer N greater than all the p-adic orders of the factors in the numerator. Then the since n is invertible mod p, it is invertible mod p^N, and we may choose a positive integer $m \geq k$ such that $mn \equiv 1 \bmod p^N$. The p-adic orders of the factors in the numerator don't change when we replace $\frac{1}{n}$ by m, and the result now follows from the fact that the ordinary binomial coefficient $\binom{m}{n}$ is an integer.
Hence, there is a power series n th root of $1+b^{-2}t$ in $R[1/b][[t]]$, where p does not divide n if the characteristic is $p > 0$, so that we can consider f as described, also in $R[1/b][[t]]$. Then, since a is divisible by every power of b in R, $af \in R[[t]]$, and so f is in the fraction field of $R[[t]]$. Since $f^n = b^n + b^{n-2}t \in R[[t]]$, f is integral over $R[[t]]$. But, as the coefficients of f include arbitrarily high negative powers with of b multiplied by scalars from the base field, we have that $f \notin R[[t]]$.

EC2. We construct a one-dimensional ring R whose local rings are all fields and DVRs such that $\operatorname{Spec}(R)$ is connected but R is not a domain.
Let H be an infinite totally ordered set with the property that between any two distinct elements of H there is another element. Let $\{x_h : h \in H\}$ be a family of indeterminates indexed by H: order them so that $x_h < x_k$ precisely when $h < k$. We form a commutative multiplicative semigroup S whose elements consist of 1 and the positive powers of the individual x_h. The multiplication is given by the rule that $x_h^m x_k^n$ is
(1) x_h^m if $h < k$ (2) x_h^{m+n} if $h = k$ (3) x_k^n if $k < h$ (forced from (1) by commutativity).
The operation is easily checked to be commutative and associative: no matter how one inserts parentheses, $x_g^m x_h^n x_k^r$ will be the least of the three variables occurring, with the exponent that is the sum of the exponents with which it occurs in the product. It is easy to verify this by considering the three cases determined by the number of occurrences of the smallest variable among the three terms.
Now let K be a field, and let R be the semigroup ring of S over K. We shall show that R has connected spectrum and is locally a domain but not a domain. We first show that R contains no idempotents except 0, 1 (thus, $\operatorname{Spec}(R)$ is connected) and no nilpotent except 0. Suppose one had such an element r. It cannot be a constant. Consider the highest degree terms that occur, and from among them pick the one with the largest variable: suppose that this term is cx_h^d, where $c \in K - \{0\}$. Then the expansion of r^2 involves a

term $c^2x_h^{2d}$ and only one such term occurs: it cannot be canceled. This term shows that $r^2 \neq 0$, and also that $r^2 \neq r$, since $\deg(r^2) > d$.
R is not a domain, since if $h < k$, $x_h = x_h x_k$ and $x_h(1-x_k) = 0$. (These relations generate the ideal of relations on the x_h, although we do not need this.) To show that R is locally a domain, we study the prime ideals of R. Call $J \subseteq H$ an *upper* (respectively, *lower*) interval if whenever $h \in J$ and $k > h$ (respectively, $k < h$) then $k \in J$ as well.
Given a prime ideal P, let J_P denote the subset of H consisting of those h such that $x_h \notin P$. Note that if $x_h \notin P$ then for all $k > h$, $x_h(1 - x_k) = 0$ implies that $1 - x_k \in P$, and so $k \in J_P$ as well. Hence, J_P is an upper interval in H. Evidently, the set I_P of h such that $x_h \in P$ is a lower interval. Note that I_P, J_P give a partition of H such that every element of I_P is less than every element of J_P. We now consider what happens when we localize at P. If k is an element of J_P but not a least element, then we can choose $x_h \notin P$ with $h < k$, and the equation $x_h(1 - x_k) = 0$ forces x_k to be identified with 1 in the localization for all k except possibly the least element in J_P. If g is any element of I_P other than the greatest, we can choose $h \in I_P$ with $g < h$, and then the equation $x_g(1 - x_h) = 0$ forces x_g to become 0 in the localization. Our assumption on the totally ordered set implies that either I_P has no greatest element, or J_P has no least element.
If there is no greatest element in I_P and no least element in J_P the localization is K. If x_h is greatest in I_P or least in J_P, all other x_k are identified with 0 or 1 after localizing at P. The local ring one gets is a localization of the polynomial ring $K[x_h]$. This shows that R is locally a domain and has Krull dimension 1. $\square$