Math 615, Winter 2019 **Problem Set #2**
Due: Friday, February 22

1. Let K be a field, let x, y be indeterminates, and let $f \in xK[[x]]$ be transcendental over $K[x]$ (e.g., over $\mathbb{C}[[x]]$ one could take f to be $e^x - 1$ or $\sin(x)$.) Let f_n be the sum of the terms of degree at most n in f. Let $R = K[x, y]$, let $m = (x, y)$, and let $(S, \mathcal{M})$ be the local ring (R_m, mR_m), so that $\widehat{S} = K[[x, y]]$. Show that in S, the ideals $I_n = (y - f_n, x^{n+1})$ are decreasing and have intersection (0), but none of them is contained $\mathcal{M}^2$. (In $\widehat{S}$, the intersection of the $I_n\widehat{S}$ is $(y - f)\widehat{S}$.) This shows that Chevalley's Lemma is not true for local rings that are not complete.

2. Let R be a universally catenary Noetherian domain, let $S \supseteq R$ be a module-finite domain extension of R, and let Q be a prime ideal of S. Let $P = Q \cap R$. Prove that P and Q have the same height. Note that R need not be normal.

3. Let R be a finitely generated $\mathbb{N}$-graded algebra over a local ring (A, m, K) with $R_0 = A$. Let J be the ideal $\bigoplus_{n=1}^{\infty} R_n$ generated by the homogeneous elements of positive degree. Let $\mathcal{M} = m + J$. Is it true that $\dim(R) = \mathrm{ht}(\mathcal{M})$? Prove your answer.

4. Assume the theorem that if (R, m) is a local ring of Krull dimension d and I is an m-primary ideal, then there is an integer-valued polynomial $H(n)$ with rational coefficients of degree d such that the length of $R/I^{n+1} = H(n)$ for all sufficiently large n.

Let $I \subseteq J$ be m-primary ideals in the local domain (R, m) of Krull dimension d. Show that if J is integral over I, the length of J^n/I^n agrees with a polynomial of degree at most $d - 1$ in n for all sufficiently large n.

5. Let $R = K[x_1, \ldots, x_m]$ and $S = K[y_1, \ldots, y_n]$ be polynomial rings over a field K. Let I be an integrally closed ideal of R and let J be an integrally closed ideal of S. Prove that $I \otimes_K J \hookrightarrow R \otimes_K S = T$ is an integrally closed ideal of T.

6. Let $I = (f_1, \ldots, f_n)R$ be an ideal of a Noetherian domain R, where the f_i are nonzero. Let B_i denote the integral closure of the ring $R[f_1/f_i, \ldots, f_n/f_i]$, and assume that each B_i is Noetherian as well. Note that $IB_i = f_iB_i$. For each i, let V_{ij} be the finitely many discrete valuation rings obtained by localizing B_i at a minimal prime of f_iB_i. Show that for every $h \in \mathbb{N}$, $g \in \overline{I^h}$ if and only if $g \in I^hV_{ij}$ for all i, j. (That is, there exist finitely many injections $R \to V_{ij}$ of R into a discrete valuation ring that may be used to test integral closure for I and every power of I.)

Extra Credit 3. Let $2 \leq r \leq s$ be integers and let x_{ij}, $1 \leq i \leq r$, $1 \leq j \leq s$, be indeterminates over a field K. Let $S = K[x_{ij} : 1 \leq i \leq r, 1 \leq j \leq s]$, let m be the ideal generated by all the x_{ij}, and let P be the ideal of $R = S_m$ generated generated by the 2×2 minors of the $r \times s$ matrix $X = \big(x_{ij}\big)$. What is the analytic spread of P?

Extra Credit 4. Let R be a normal local domain of dimension 2 and let P be a height one prime ideal of R. Let S be the *symbolic Rees algebra* of P, that is, the subring of the polynomial ring $R[t]$ spanned by all the elements $P^{(n)}t^n$ for $n \in \mathbb{N}$. Here, $P^{(n)} = P^nR_P \cap R$. One may also describe S as $R + Pt + P^{(2)}t^2 + \cdots + P^{(n)}t^n + \cdots$. Suppose that S is finitely generated over R. Prove that for some integer $n \geq 1$, $P^{(n)}$ is principal.