Math 615, Fall 2019 **Problem Set #2: Solutions**

1. In $\widehat{S} = K[[x,y]]$, we have that $I_n = (y - f, x^{n+1}) \subseteq (y - f) + \widehat{m}^n$, where $\widehat{m} = (x,y)K[[x,y]]$. Since ideals of a complete local ring are $\widehat{m}$-adically closed, $\bigcap_n (I_n K[[x,y]])$ is $(y-f)K[[x,y]]$, which must contain the intersection of the I_n in S. But $(y-f)K[[x,y]] \cap S = (0)$. To see this, note that we can clear denominators and so, if the intersection is nonzero, we can find a nonzero polynomial $G(x,y) \in K[x,y]$, that is a multiple of $(y - f)$. There is a homomorphism $K[[x,y]] \to K[[x]]$ that is the identity on $K[[x]]$ and sends $y \mapsto f$. If $G = H(y - f)$, then $G \mapsto 0$ under this homomorphism, i.e., $G(x,f) = 0$. This contradicts the transcendence of f over $K(x)$. It is clear that $y - f \in I_n - m^2$.

2. Since R is universally catenary and S is module-finite over R, which implies finitely generated as an R-algebra, we may apply the dimension formula. Since S is module-finite over R, we also have that S/Q is module-finite over R/P. It follows that both transcendence degrees are 0, and so $\mathrm{height}\,(Q) - \mathrm{height}\,(P) = 0$, as required.

3. We shall prove by Noetherian induction that the homogeneous maximal ideal always has the largest height: that is we kill a maximal element of the set of homogeneous ideals I such that R/I is a counterexample. Thus, we may assume the result is true for any quotient by a homogeneous ideal. Suppose that the homogeneous maximal ideal is $\mathcal{M}$ and $\mathcal{Q}$ is another maximal ideal of greater height. Since any maximal chain of primes descending from $\mathcal{Q}$ ends in a minimal prime, which must be homogeneous, we still have a counter-example after killing one of the homogeneous minimal primes. Thus, we may assume without loss of generality that the ring is a domain. If $\mathcal{Q}$ contains a nonzero element $a \in A$ (which must be a non-unit and so in m) then $a \in \mathcal{Q}$ and $a \in \mathcal{M}$. When we kill aR, both heights drop by one, and we still have a counterexample. Therefore, we may assume that $\mathcal{Q}$ does not meet A. Thus, Q corresponds to a prime ideal of $S = (A - 0)^{-1}R$, which is an $\mathbb{N}$-graded algebra with $S_0 = L = \mathrm{frac}\,(A)$. Hence, $\mathcal{Q}$ has the same height as the unique homogeneous maximal ideal of S, which is JS. But since the prime J of R is contained in $\mathcal{M} = m + J$, we then have $\mathrm{height}\,(\mathcal{Q}) \le \mathrm{height}\,(\mathcal{M})$. $\square$

4. Since $\ell(J^n/I^n) = \ell(R/I^n) - \ell(R/J^n)$, it has polynomial behavior for $n \gg 0$. There exists a nonzero element c of R such that $cJ^n \in I^n$ for all n, i.e., c kills J^n/I^n. If c is a unit, $J^n/I^n = 0$, and we assume otherwise. From the exact sequence $0 \to \mathrm{Ann}_{R/I^n} c \to R/I^n \xrightarrow{c\cdot} R/I^n \to (R/I^n + cR) \to 0$, we have that the alternating sum of the lengths of these modules is 0. The middle terms cancel, and it follows that $\ell(J^n/I^n) \le \ell(\mathrm{Ann}_{R/I^n} c) = \ell(R/(I^n + cR)$. We can rewrite the rightmost term as $\ell(\overline{R}/\mathfrak{A}^n)$, where $\overline{R} = R/cR$ and $\mathfrak{A} = I\overline{R}$. This length is the Hilbert function of $I\overline{R}$, and $\dim(\overline{R}) < d$. $\square$

5. Since $I \hookrightarrow R$ and S is K-flat, $I \otimes_K S \hookrightarrow R \otimes_K S = T$ and may be identified with IT. IT is integrally closed by a previous exercise, since T is a polynomial ring over R. Similarly, $R \otimes_K J \hookrightarrow T$ may be identified with JT and is integrally closed. But the image $I \otimes_K J$ in T is the intersection of the images of $I \otimes_K S$ and $R \otimes_K J$ (this is simply a fact about K-vector spaces), and since the intersection of two (or any number) of integrally closed ideals is integrally closed, it follows that the image of $I \otimes J$ in T is an integrally closed ideal.

6. We need to show that $u \notin \overline{I^n}$ implies that $u \notin I^n V_{ij}$ for some choice of i, j. (The other direction is obvious.) But if $u \notin \overline{I^n}$ we can choose a map $R \hookrightarrow V$ for some DVR V such that $u \notin I^n V$. Now IV is principal, and one of the f_i generates. Hence, for some choice of i, $R[f_1/f_i, \dots, f_n/f_i] \subseteq V$. Since V is integrally closed, it follows that for this choice of i, $B_i \subseteq V$. Since $y \notin I^n V$, we have $u \notin I^n B_i = f_i^n B_i$. Since B_i is a normal Noetherian ring, the associated primes of $f_i^n B_i$ are the same as the minimal primes of $f_i^n B_i$, and these are the same as the minimal primes of $f_i B_i$. We can choose one of these, say P_{ij}, such that u is not in the P_{ij}-primary component of f_i^n, i.e., $u \notin f_i^n (B_i)_{P_{ij}} \cap B_i$. In particular, $u \notin f_i^n V_{ij} = I^n V_{ij}$. $\square$

EC3. The analytic spread will be the same as the dimension of the ring B generated by the minors (which always means 2×2 minors here), by a class theorem. If $r = 2$ or $s = 2$ and the other is at least two, the dimension of the ring generated by the minors is $rs - 3$. To see this, we may assume without loss of generality that $r = 2 \leq s$: we must show that the dimension is $2s - 3$, and this is the same as the transcendence degree of the field generated by the minors. Consider the $2s - 3$ minors involving the first or second column or both. We shall show that these are algebraically independent and generate the fraction field of B. To see independence, specialize x_{11} to 1 and x_{21} and x_{12} to 0. The minors then yield, up to sign, x_{2j} for $j \geq 2$ and $x_{22} x_{1j}$ for $j \geq 3$, which generate the same field as the $2s - 3$ (unspecialized) variables in the matrix. Since the specified minors are algebraically independent even after specialization, they are algebraically independent. Let M_{ij} denote the minor determined by the i, j columns for $i < j$. For $2 < i < j$, a straightforward calculation shows that one has the relation $M_{12} M_{ij} - M_{1i} M_{2j} + M_{1j} M_{2i} = 0$, and one can solve for M_{ij} in terms of minors involving the first or second column. Hence, every M_{ij} is in the field $K(M_{1i}, M_{2j} : i, j)$. $\square$

If $r, s \geq 3$ the dimension of B is rs. To see this it suffices to show that the field F generated by all the variables is algebraic over the field generated by the minors. Since each variable occurs in some 3×3 submatrix, it suffices to prove this when $r = s = 3$. Let X^* denote classical adjoint of X, the transpose of the cofactor matrix. Note that the entries of X^* are the 2×2 minors of X, up to sign. Let $D = \det(X)$. Then $XX^* = DI$. Taking determinants, we have $D \det(X^*) = D^3$, and $\det(X^*) = D^2 \in F$. Thus, D is algebraic over F. But $X^{-1} = (1/D) X^*$ has entries in $F[D]$, and then $X = (X^{-1})^{-1}$ has entries in $F[D]$ as well. Thus, all the variables are algebraic over F. $\square$

EC4. Assuming that the algebra is finitely generated we can assume that $R + P^{(n)} t^n + P^{(2n)} t^{2n} + \cdots$ is generated by $P^{(n)} t^n$ for some integer n. This implies that $(P^{(n)})^k = P^{(nk)}$ for all $k \geq 1$. Let $q = P^{(n)}$. Then $S = \mathrm{gr}_q R = R/q + q/q^2 + \cdots +$, and every element of R/q is a nonzerodivisor on S (since the q^k are *symbolic powers* of P) over the one-dimensional ring R/q. Thus, if we kill a nonzero element in the maximal ideal of R/q, the dimension of the ring must drop, and is at most one. It follows that the Krull dimension of $R/m \otimes_R S$ is at most one. But this means that the analytic spread of q is 1. If the field is infinite, q is the integral closure of a principal ideal. If the field is finite, there will still be a homogeneous parameter in $R/m \otimes_R S$ in some sufficiently high degree, and from this we can conclude in this case as well that some power of q is the integral closure of a principal ideal and, therefore, principal, since R is normal. $\square$