Math 615, Fall 2019 **Problem Set #3: Solutions**

1. R is the K-span of all monomials of even degree and those of odd degree that are multiples of x_i^3 for some i. It follows that the x_i^3 and $x_1^2 \cdots x_n^2$ are in the conductor, and generate it: if say x_n does not occur in $\mu = x_1^{a_1} \cdots x_{n-1}^{a_{n-1}} x_n^{a_n}$ with all $a_j \leq 2$ and some $a_i < 2$, then μ has even degree and $x_i\mu \notin R$.

2. We map $R[X_1, \ldots, X_n] \twoheadrightarrow S$ by $X_i \mapsto x_i$ all i. If $\mu = \mu(X)$ is any monomial of degree d in the X_j, then $\mu(X) \mapsto \mu(x) \in R$, and the differences $\mu(X) - \mu(x)$ generate the kernel, since killing them is sufficient to identify the image of R with the K-isomorphic subring of $R[X_1, \ldots, X_n]$ generated by monomials of degree d in the X_j. The elements $\mu(x)$ are all killed by the $\partial/\partial X_j$, and so we obtain the matrix $(\partial\mu/\partial X_j)$ as μ runs through all monomials of degree d in the X_j. When we take size n minors and evaluate, we get elements of degree $n(d-1)$ in the x_j. Hence, $\mathcal{J}_{S/R}$ is contained in $(x_1, \ldots, x_n)^{n(d-1)}S$. We show that they are equal: we obtain every monomial as a Jacobian minor. After renumbering the variables we may assume the monomial has the form $\mu(x) = x_1^{a_1} \cdots x_n^{a_n}$ where we have that $a_1 \geq a_2 \geq \cdots \geq a_n$. From these inequalities, $a_1 \geq d-1$, $a_1 + a_2 \geq 2(d-1)$, and, in general, $a_1 + \cdots + a_i \geq i(d-1)$, $1 \leq i \leq n$. It follows we can factor $\mu(X)$ in the form $\mu_1 \cdots \mu_n$, all of degree $d-1$, where μ_1 involves only X_1, μ_2 involves only X_1 and X_2, and so forth: μ_i involves only $X_1, \ldots, X_i$, $1 \leq i \leq n$. (In choosing μ_j once μ_i for $i < j$ has been picked, use the highest power of the lowest numbered variable available before using any higher numbered variables.) Then the Jacobian submatrix corresponding to the partial derivatives of $\mu_1 X_1, \mu_2 X_2, \ldots, \mu_i X_i, \ldots, \mu_n X_n$ is upper triangular, and the diagonal entries have the form $b_1\mu_1, b_2\mu_2, \cdots, b_n\mu_n$, where for every i, $1 \leq b_i \leq d$, so that the b_i are invertible in K. Hence, the image of the corresponding Jacobian determinant in S is a unit times $\mu(x)$. $\square$

3. Since $x \mapsto u$ has order 1, T_1 is the localization of $R[y/x]$ at the prime ideal lying under uV. The map of $\mathbb{R}[x, y/x] \to V$ sends $x \mapsto u$ and $y/x \mapsto zt$, which is a unit of V. The image of $\mathbb{R}[x, y/x]$ in the residue class field $\mathbb{C}(t)$ of V is $\mathbb{R}(it)$, where i is the image of z. Since this is transcendental over $\mathbb{R}$, the dimension formula implies that T_1 has dimension 1, and must be the localization of $\mathbb{R}[x, y/x]$ at $xR[x, y/x]$. The residue class field of T_1 may be identified with $\mathbb{R}(it) \subseteq \mathbb{C}(t)$. Since T_1 has dimension one, the quadratic sequence ends with T_1.

4. Let $D = R[x_2/x_1, \ldots, x_n/x_1] = \bigcup_{n=0}^{\infty} m^n/x_1^n$. T_1 is D localized at the prime ideal of elements of positive m-adic order. This ideal may also be described as $\bigcup_{n=0}^{\infty} m^n/x_1^n$, and is generated by x_1. Killing x_1 in D gives $K[y_2, \ldots, y_n]$, a polynomial ring, by a class theorem, where y_j is the image of x_j/x_1 for $j \geq 2$, and so the residue class field of the first transform is $K(y_2, \ldots, y_d)$. By the dimension formula, $\dim(T_1) = 1$ and the sequence ends with T_1, which must be the same as the valuation ring of the fraction field corresponding to the m-adic order valuation, and is essentially of finite type over R.

5. After each transform, we still have local maps $R \subseteq T_i \subseteq V$, and if K_i is the residue class field of T_i, we have $K \subseteq K_i \subseteq K$, so that the extension of residue class fields from R to T_i is an isomorphism. It follows that the Krull dimension of every T_i is 2, and so the sequence of quadratic transforms does not terminate.

6. Let $t_1, \ldots, t_n$ be indeterminates over R and let $S = R[I_1t_1, \ldots, I_nt_n] \subseteq R[t_1, \ldots, t_n]$. This domain is finitely generated over K (choose finitely many generators f_{ij} for each I_j and use the the $f_{ij}t_j$ as generators of S). The normalization S' of this ring is contained in $R[t_1, \ldots, t_n]$ and, by a class theorem, graded by the monomials in the t_j: hence, it has the form $\sum_{\underline{i}} J_{\underline{i}}t^{\underline{i}}$ where $\underline{i}$ runs through elements $(i_1, \ldots, i_n)$ of $\mathbb{N}^n$. S' is module-finite over S by a class theorem. (What we need about R is that the normalization of a domain D finitely generated over R is module-finite over D: this also holds when R is complete local or, more generally, excellent.) If one chooses a homogeneous equation of integral dependence for $rt^{\underline{i}}$ one sees that the condition for integral dependence is precisely that $r \in \overline{I_{\underline{i}}}$. This means that $J_{\underline{i}} = \overline{I^{\underline{i}}}$. The stated condition now follows easily if c is the maximum of the degrees of finitely many homogeneous S-module generators for S'.

EC5. $K \to S$ splits as a map of vector spaces: let $\theta : S \to K$ be a splitting. It follows that $R \to R \otimes_K S$ splits as a map of R-modules, using $\mathrm{id}_R \otimes \theta$ as the splitting. This implies that R is normal. The proof for S is similar.

$\mathfrak{C} \otimes_K \mathfrak{D}$ is an ideal of both $A \otimes_K B$ and of $R \otimes_K S$. We shall show that it is the conductor J. Since $\mathfrak{C} \otimes_K \mathfrak{D}$ is the intersection of $\mathfrak{C} \otimes_K S$ and $R \otimes_K \mathfrak{D}$, it suffices to show that J is contained in the intersection of these. By symmetry, it suffices to show that J is contained in $\mathfrak{C} \otimes_K S$. Choose a K-basis h_ν for S. Then $R \otimes_K S$ is a free R-module on the basis $1 \otimes h_\nu$, $A \otimes_K S$ is a free A-module on the basis $1 \otimes h_\nu$. Suppose that $\sum_{j=1}^t r_j \otimes h_j$ is in J. Then for every $r \in R$, $\sum_{j=1}^t rr_j \otimes h_j$ is in $A \otimes_K B \subseteq A \otimes_R S$, and this is only true if all the $rr_j \in A$. But if every $rr_j \in A$, then all the $r_j \in \mathfrak{C}$, as required. $\square$

EC6. It suffices to show that if $a, b \in m - \{0\}$ then $a|b$ or $b|a$. We follow the suggested plan. Since principal ideals are integrally closed, R is normal. It follows that $R[t]$ and its localization $R(t)$ are normal. By a calculation in class, every a^ib^{n-i} is in $\overline{(a^n, b^n)R}$ and, hence, in $(a^n, b^n)R$. Thus, with $I = (a, b)R$ we have $I^n = (a^n, b^n)R$, and this continues to hold when we expand to $R(t)$. It follows that the length of $I^nR(t)/mI^nR(t)$ is at most two, and so the Hilbert polynomial of $K(t) \otimes_{R(t)} \mathrm{gr}_{IR(t)}R(t)$ is a constant. t follows that the Krull dimension of this ring is 1, and so the analytic spread of $IR(t)$ is one, and $IR(t)$ is integral over a principal ideal. Since $R(t)$ is normal, principal ideals are integrally closed, and, therefore, $(a, b)R(t)$ is principal. By Nakayama's lemma, either a or b is a generator: suppose a is. Then $b \in aR(t)$. Since $R(t)$ is faithfully flat over R, this impilies $b \in aR$. $\square$