Math 615, Winter 2019 **Problem Set #4**

Due: Monday, April 8

1. Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a field K, let $t \geq 1$ be an integer, and let S denote the K-algebra $K[x_1, x_2/x_1^t, \dots, x_n/x_1^t]$. Determine $\mathcal{J}_{S/R}$ (the result was stated in class when $t = 1$) and $W_{S/R}$.

2. Let $n \geq 1$ be an integer, let $R = K[x, y]$ and let $S = K[X^4, X^3Y, XY^3, Y^4]$, and think of R as a subring of S using the K-algebra injection such that $x \mapsto X^4$ and $y \mapsto Y^4$. Assume that $\mathrm{char}(K) \neq 2$. Let S' denote the integral closure of S. Determine $W_{S/R}$ and $W_{S'/R}$ as submodules of $\mathcal{L} = \mathrm{frac}(S)$.

3. Let S_1 and S_2 be finitely generated, torsion-free, and generically étale extensions of polynomial rings $R_1 = K[x_1, \dots, x_m]$ and $R_2 = K[y_1, \dots, y_n]$, respectively, let $S = S_1 \otimes_K S_2$, and let $R = R_1 \otimes_K R_2$. Show that S is finitely generated, torsion-free, and generically étale over R.

4. With the same hypothesis as in **3.**, determine whether $W_{S/R}$ can be identified with $W_{S_1/R_1} \otimes_K W_{S_2/R_2}$.

5. Let K be a field of prime characteristic $p > 0$ with $p \neq 3$, let $S = K[X, Y, Z]$ be a polynomial ring over K, let $R = K[X, Y, Z]/(X^3+Y^3+Z^3) = K[x, y, z]$ and let $I = (x, y)R$. Prove that $z^2 \in I^*$ but that $z \notin I^*$. (Since this holds for all $K = \mathbb{Z}/p\mathbb{Z}$, $p \neq 3$, it follows that $z^2 \in I^*$ when K is allowed to be $\mathbb{Q}$ as well.)

6. Let R be a Noetherian ring of prime characteristic $p > 0$. Let I be an ideal such that $I = I^*$, let $f \in R$ and $J = I :_R fR$. Must $J = J^*$? Prove your answer.

Extra Credit 7. Let R be a regular Noetherian domain of characteristic $p > 0$ and let S be a module-finite domain extension of R such that the extension of fraction fields is separable. If D is domain of characteristic p, let $D^{1/q} = \{d^{1/q} : d \in D\}$. Show that $\mathcal{J}_{S/R}S^{1/q} \subseteq R^{1/q}[S]$ for all q.

Extra Credit 8. Let M and N be finitely generated modules over a Noetherian ring R. Give as an informative characterization as you can of the associated primes of $\mathrm{Hom}_R(M,\, N)$.