Math 615, Fall 2019 **Problem Set #4: Solutions**

1. By a theorem in the notes, $S \cong R[Y_2, \ldots, Y_n]/I$ where $I = (g_1, \ldots, g_{n-1})$ if we take $g_i = x_1^t Y_{i+1} - x_{i+1}$ for $i \geq 1$. Since each of the quotients $R[Y_2, \ldots, Y_n]/(g_1, \ldots, g_i)$ is an integral domain, namely $R[x_2/x_1, \ldots, x_{i+1}/x_1][Y_{i+2}, \ldots, Y_n]$, $g_1, \ldots, g_n$ is a regular sequence and it is clearly special: there is only one prime, which it generates even without localization. $\mathcal{J}_{S/R}$ is generated by the determinant of a size $n-1$ upper triangular matrix in which all the diagonal entries are x_1^t. Thus, $\mathcal{J}_{S/R} = x_1^{t(n-1)}R$, and $W \subseteq \mathrm{frac}\,(R)$ is the cyclic S-module generated by $1/x_1^{t(n-1)}$.

2. We can write $S = R[Z_1, Z_2]/I$ where Z_1, Z_2 map to X^3Y, XY^3 respectively. $X^4 = x$ and $Y^4 = y$ are already in R. Note that I contains the elements $g_1 = Z_1^3 - x^2Z_2$, $g_2 = Z_2^3 - y^2Z_2$ and $f = Z_1Z_2 - xy$, $g = yZ_1^2 - xZ_2^2$. S has a K-basis consisting of all monomials in X, Y of total degree a multiple of 4 except X^2Y^2. Setting $g_1 = g_2 = f = g = 0$ provides sufficient relations to write any monomial in S as $x^iy^jZ_1^a$ where $0 \leq a \leq 2$ or $x^iy^jZ_2^b$ where $i = 0$ and $b \leq 2$ or $i > 0$ and $b \leq 1$ (the image of these is a K-basis for S). Hence, g_1, g_2, f, g generate I. Moreover, $g_1 = Z_1^3 - x^2Z_2$ and $g_2 = Z_2^3 - y^2Z_1$ are special: they are clearly a regular sequence, since they are monic in distinct variables, and the Jacobian determinant is the image of $(3Z_1^2)(3Z_2^2) - (-x^2)(-y^2) = 9x^2y^2 - x^2y^2 = 8x^2y^2$, which is not 0. Thus, $W_{S/R}$ is the image in $L = \mathrm{frac}\,(S)$ of $(g_1, g_2) : I = (g_1, g_2) : (f, g)$ under the map sending $u \mapsto \overline{u}/8(xy)^2$, where $\overline{u}$ is the image of u in L. First note that $K[x, y, Z_1, Z_2]/(g_1, g_2)$ is free over $K[x, y]$ and x, y are not zerodivisors modulo (g_1, g_2). We shall show that $(g_1, g_2) : I = (F, G, H) + (g_{1,2}\,)$ where $F = Z_1^2Z_2 + xyZ_1, G = Z_1Z_2^2 + xyZ_2$, and $H = yZ_1^2 + xZ_2^2$. (Note that $Ff = Z_2^2g_1 + x^2G_2, Fg = Z_1Z_2yg_1 - Z_1^2xg_2$, that $Gf, Gg \in (g_1, g_2)$ by symmetry, $Hf = yZ_2g_1 + XZ_1G_2$, and $Hg = y^2Z_1g_1 - x^2Z_2g_2$.) To see that no other element Q is in the colon, note that by subtracting off multiples of these elements and g_1, g_2 we can assume that such an element $aZ_1^2 + bZ_2^2 + cZ_1Z_2 + dZ_1 + eZ_2 + f$ where the lower coefficients are in $K[x, y]$ is at worst quadratic in Z_1, Z_2. Moreover by subtracting off a multiple of G we can assume that the coefficient of Z_2^2 in $K[x, y]$ contains no term involving x. We may also assume that the element is homogeneous in the $\mathbb{N}^2$ grading on $K[x, y, Z_1, Z_2]$ such that the degrees of x, y, Z_1, Z_2 are $(4, 0), (0, 4), (3, 1)$, and $(1, 3)$, respectively. The leading term of this element must multiply the leading terms of f into (Z_1^3, Z_2^3), and it must be in Z_1^2, Z_1Z_2, Z_2^2. Hence, at least one of a, b, c is not 0.
If $b \neq 0$ has degree $(0, 4t)$, then all terms of Q have degree $(2, 4t + 6)$. This forces $a = c = d = e = f = 0$, and the element has the form $y^{4t}Z_2^2$. Since y is not a zerodivisor on (g_1, g_2), this implies $Z_2^2 \in (g_1, g_2) : I$. But $Z_2^2f = Z_1Z_2^3 - xyZ_2^2 = Z_1g_2 + y^2Z_1Z_2 - xyZ_2^2 \notin (g_1, g_2)$, since $y^2Z_1Z_2 - xyZ_2^2 \notin (g_1, g_2)$ (its term of highest degree in Z_1, Z_2 is not in (Z_1^3, Z_2^3).)
Hence, $b = 0$. If $a \neq 0$ has degree $(4s, 4t)$ the first term in Q has degree $(4s+6, 4s+2)$. This forces $b = c = e = f = 0$. This would mean that $x^sy^tZ_1^2$ is in the colon, and that would force Z_1^2 in $(g_1, g_2) : I$, which is false by symmetry. Finally if $a = b = 0$, then working mod 4 the bidegree of cZ_1Z_2 is (0,0). This forces $d = e = 0$. So the only remaining possibility is $x^sy^tZ_1Z_2 + \alpha x^{s+1}y^{s+1}$, where $\alpha \in K$. Since x, y are nonzero divisors mod (g_1, g_2), this would imply $Z_1Z_2 + \alpha xy \in (g_1, g_2) : I$. But $(Z_1Z_2 - \alpha xy)f$ has leading term (with respect to degree in Z_1, Z_2) $Z_1^2Z_2^2 \notin (Z_1^3, Z_2^3)$, which eliminates this possibility. Thus, $(g_1, g_2) : I$ is as described. Taking images in L and dividing by the Jacobian $8x^2y^2$, we obtain three

generators: $\big((X^3Y)^2(XY^3) + X^4Y^4(X^3Y)\big)/8X^8Y^8 = X^{-1}Y^{-3}/8$ from F, $X^{-3}Y^{-1}$, by symmetry, from G, and $\big(Y^4(X^3Y)^2 + X^4(XY^3)^2)\big)/8X^8Y^8 = X^{-2}Y^{-2}/4$ from H. Since two is invertible in the field, $W_{S/R} = X^{-3}Y^{-1}S + X^{-2}Y^{-2}S + X^{-1}Y^{-3}S \subseteq \mathrm{frac}\,(S)$.

For S' we introduce a third generator, $Z_3 \mapsto x^2y^2$ over $K[x,y]$. We get six new generators for the ideal of relations: these are 2×2 minors of the matrix $\begin{pmatrix} x & Z_1 & Z_3 & Z_2 \\ Z_1 & Z_3 & Z_2 & y \end{pmatrix}$. Once we map to S', the image of the second row is Y/X times the first row. S' is normal: it is a direct summand of $K[X,Y]$ as an S'-module (the complement is spanned over K by all monomials of degree not divisible by 4). Since S' is normal, $W_{S'/R} = S' :_L \mathcal{J}_{S/R}$. These six elements, (#) $\quad Z_1^2 - xZ_3, xZ_2 - Z_1Z_3, Z_1Z_2 - xy, Z_3^2 - Z_1Z_2, Z_2Z_3 - yZ_1, Z_2^2 - yZ_3$ generated $\mathrm{Ker}\,(R[Z_1, Z_2, Z_3] \twoheadrightarrow S'$: a basis for S' over K consists of all monomials $x^iy^jZ_m^a$ where $0 \le a \le 1$ and m is 1, 2, or 3, and the relations one gets by setting the elements (#) equal to 0, which imply $Z_3^2 - xy = 0$ as well, can be used to show that every monomial in S' is congruent to a basis element. We replace $Z_3^2 - Z_1Z_2$ by its sum with $Z_1Z_2 - xy$, i.e, with $Z_3^2 - xy$. We then get a 3×6 Jacobian matrix, $\begin{pmatrix} 2Z_1 & -Z_3 & Z_2 & 0 & -y & 0 \\ 0 & x & Z_1 & 0 & Z_3 & 2Z_2 \\ -x & -Z_1 & 0 & 2Z_3 & Z_2 & -y \end{pmatrix}$. Since 2 is invertible in K, it is straightforward to see that the ideal generated by the images of size 3 minors is generated by the elements $X^3Y^9, X^4Y^8, \ldots, X^iY^{12-i}, \ldots, X^8Y^4, X^9Y^3$. It follows that $W_{S'/R}$ is spanned *over* S' by the elements $X^{-3}Y^{-1}, X^{-2}Y^{-2}, X^{-1}Y^{-3}$. By a class theorem, $W_{S'/R} \subseteq W_{S/R}$, and so in this example they are the same.

3. Quite generally if M_i is torsion-free over the domain R_i for $i = 1, 2$ and $R_3 = R_1 \otimes_K R_2$ is a domain, then $M_1 \otimes M_2$ is torsion-free over R_3. (This is true even when K need not be a field if, say, M_1, R_2 are flat over K.) Since each M_i is a directed union of finitely generated submodules it suffices to do the case where the M_i are finitely generated modules. Since M_i is torsion-free, it embeds in G_i, a finitely generated free R_i-module. Then $M_1 \otimes_K M_2 \hookrightarrow M_1 \otimes_K G_2 \hookrightarrow G_1 \otimes_K G_2$, and the latter is free over R_3. If S_i is generated over R_i by $s_{i,1}, \ldots, s_{i,r_i}$ the elements $s_{1,j} \otimes 1$ and $1 \otimes s_{2,k}$ generate $S_1 \otimes S_2$ over R_3. Finally, let $\mathcal{K}_i$ be the fraction field of R_i for $i = 1, 2, 3$. Note that $L = \mathcal{K}_1 \otimes_K \mathcal{K}_2 \subseteq \mathcal{K}_3$: L is not a field, but its fraction field is $\mathcal{K}_3$. Then, by hypothesis, $cK_i \otimes S_i$ is a finite product of separable field extensions $\mathcal{F}_{ij}$ of $\mathcal{K}_i$. We have in consequence that $\mathcal{K}_3 \otimes R_3(S_1 \otimes_K S_2) \cong \mathcal{K}_3 \otimes_L \big(L \otimes_{R_3} \otimes (S_1 \otimes_K S_2)\big) \cong \mathcal{K}_3 \otimes_L \big((\mathcal{K}_1 \otimes_{R_1} S_1) \otimes_K (\mathcal{K}_2 \otimes_{R_2} S_2)\big) \cong \mathcal{K}_3 \otimes_L \big((\prod_{j=1}^a \mathcal{F}_{1j}) \otimes_K (\prod_{k=1}^b \mathcal{F}_{1k})\big) \cong \prod_{j,k}(\mathcal{K}_3 \otimes_L (\mathcal{F}_{1j} \otimes_K \mathcal{F}_{2j})$. Therefore, it suffices to show that each $B_{jk} = \mathcal{K}_3 \otimes_L (\mathcal{F}_{1j} \otimes_K \mathcal{F}_{2k})$ is a finite product of finite separable field extensions of $\mathcal{K}_3$. Now each of $\mathcal{F}_{1,j}$, $\mathcal{F}_{2,k}$ has the form $\mathcal{K}_i[Z_i]/(F_i)$, $i = 1, 2$, where F_i is a separable monic polynomial in Z_i over $\mathcal{K}_i$. We have $B_{jk} \cong \mathcal{K}_3[Z_1, Z_2]/(F_1, F_2)$. Since F_1 is a separable polynomial, $\mathcal{K}_3[Z_1]/(F_1)$ is a product of separable field extensions $\mathcal{G}_j$ of $\mathcal{K}_3$, and for every j. $\mathcal{G}_j[Z_2]/(F_2)$ is a finite product of separable field extensions of $\mathcal{G}_j$. Thus, the extension is generically étale.

4. This identification can be made. If we have a presentation $S_1 = T_1/I_1$ with special sequence $g_1, \ldots, g_n$ and $S_2 = T_2/I_2$ with special sequence $h_1, \ldots, h_r$, then it is straightforward to see that $S_3 = S_1 \otimes_K S_2 = T_3/(I_1 \otimes T_2 + T_1 \otimes I_1)$, with $T_3 = T_1 \otimes_K T_2$. Also $g_1, \ldots, g_n, h_1, \ldots, h_r$ is a special sequence. Then W_{S_3/R_3} may be computed using the map that takes each element of $((\underline{g},\underline{h}) :_{T_3} (I_1+I_2))/(\underline{g},\underline{h})$ represented by w in the numerator ideal to w/γ, and the Jacobian determinant gamma is easily verified to be $\gamma_1\gamma_2$, where γ_1, γ_2 are the Jacobians associated with $\underline{g}$ and $\underline{h}$, respectively. Now $(\underline{g},\underline{h}) :_{T_3} (I_1+I_2) = \bigl((\underline{g},\underline{h}) :_{T_3} I_1\bigr) \cap \bigl(\underline{g},\underline{h}) :_{T_3} I_2\bigr)$. The first term may be identified with $(\underline{g}) :_{T_1} I_1 \otimes T_2 + T_1 \otimes_K (\underline{h})$ (this is an instance of the class result that colons of finitely generated ideals commute with flat base change, since we may work mod $(\underline{h})T_2$ and $T_1 \otimes_K (T_2/(\underline{h})T_2)$ is flat over T_1), and the second with $T_1 \otimes_K (\underline{h}) :_{T_2} I_2 + (\underline{g}) \otimes_K T_2$. Hence, the intersection may be identified with $\bigl((\underline{g}) :_{T_1} I_1\bigr) \otimes_K \bigl((\underline{h}) :_{T_2} I_2\bigr)$. Thus, $\bigl((\underline{g},\underline{h}) : (I_1+I_2)\bigr)/(\underline{g},\underline{h}) \cong \bigl((\underline{g}) :_{T_1} I_1/(\underline{g})\bigr) \otimes_K \bigl((\underline{h}) :_{T_2} I_2/(\underline{h})\bigr)$, and $W_{S/R} \cong W_{S_1/R_1} \otimes_K W_{S_2/R_2}$ follows from the identification of γ with $\gamma_1\gamma_2$: note that $(\overline{u} \otimes \overline{v})/\gamma = (\overline{u}/\gamma_1) \otimes (\overline{v}/\gamma_2)$. $\square$

5. Note that mod the sum of the cubes, every element can be represented uniquely in the form $f + gz + hz^2$ where $f, g, h \in K[x, y]$. If we take $c = z$, to show that $z^2 \in I^*$ it suffices to show that $z(z^2)^q \in (x^{2q}, y^{2q})$ for all q. Let a be the remainder when $2q+1$ is divided by 3, and write $2q + 1 = 3h + a$. Then $cz^q = z^{2q+1} = (z^3)^h z^a$, and it suffices to show that $(z^3)^h \in (x^q, y^q)$, i.e. that $(x^3 + y^3)^h \in (x^q, y^q)$. When the left hand side is expanded, every term has the form $x^s y^t$ where $s + t = 3h = 2q + 1 - a$ where $0 \le a \le 2$. Hence, $s + t \ge 2q - 1$, and one of s, t is at least q.

I will leave the problem of showing that $z \notin I^*$ as a continuing problem.

6. Yes, $J = J^*$. Let $r \in J^*$. Then there exists an element $c \in R$ not in any minimal prime such that $cu^q \in J^{[q]}$ for all $q \gg 0$. If $cu^q \in J^{[q]} = (I :_R fR)^{[q]}$, it is a sum $\sum_{i=1}^h r u_i^q$, where each $u_i \in I :_R fR$. Then $fu_i \in I$, and so $f^q u_i^q \in I^{[q]}$ for all i. It follows that $f^q(cu^q) \in I^{[q]}$, i.e., $c(fu)^q \in I^{[q]}$ for all $q \gg 0$. Hence, $fu \in I^* = I$, and $u \in I :_R fR = J$. $\square$

EC7. Since $R^{1/q}$ is faithfully flat over R, $B = R^{1/q} \otimes_R S$ is faithfully flat over S. Hence, elements of $S - \{\{0\}$ and of $R - \{0\}$ are nonzerodivisors on B, and if tensor with $\mathcal{K} = \mathrm{frac}\,(R)$ we obtain that $R^{1/q} \otimes_R S \subseteq \mathcal{K} \otimes_R (R^{1/q} \otimes_R S) \cong (\mathcal{K} \otimes_R \mathcal{K}) \otimes_R (R^{1/q} \otimes_R S) \cong$ (by the associativity of tensor) $(\mathcal{K} \otimes_R R^{1/q}) \otimes_R (\mathcal{K} \otimes_R S) \cong \mathcal{K}^{1/q} \otimes_R \mathcal{L}$, where $\mathcal{L} = (L)$. (With an integral extension domain D of a domain R, inverting every element of $R - \{0\}$ inverts every element of $D - \{0\}$, since each nonzero element of D has a nonzero multiple in R.) The tensor product of two modules over $W^{-1}R$ is the same whether the base is taken to be R or $W^{-1}R$, so the last term becomes $\mathcal{K}^{1/q} \otimes_{\mathcal{K}} \mathcal{L}$. Because L is separable over $\mathcal{K}$, $\mathcal{L} \cong \mathcal{K}[X]/(F)$ where F is a separable monic polynomial in X, and the tensor product is $\mathcal{K}^{1/q}[X]/(F)$, which is reduced. Thus, $\mathcal{K}^{1/q} \otimes L$ is reduced. Every element u has the form $\sum_{j=1}^t \alpha_j^{1/q} \otimes \lambda_j$ with the $\alpha_j \in \mathcal{K}$ and the $\lambda_j \in \mathcal{L}$. Then $u^q \sum_{j=1} t\alpha \otimes \lambda^q = 1 \otimes \sum_{j=1}^t \alpha_j \otimes \lambda^q \in \mathcal{L}$, and so is a unit if it is not zero. Since $\mathcal{K}^{1/q} \otimes_{\mathcal{K}} \mathcal{L}$ is reduced, every nonzero element u has a nonzero power in $\mathcal{L}$ that is a unit, and so every nonzero element is a unit. Thus, $\mathcal{K}^{1/q} \otimes_{\mathcal{K}} \mathcal{L}$ is a field, and the map $\mathcal{K}^{1/q} \otimes_{\mathcal{K}} \mathcal{L} \to \mathcal{L}^{1/q}$ induced by $\mathcal{K}^{1/q} \subseteq \mathcal{L}^{1/q}$ and $\mathcal{L} \subseteq \mathcal{L}^{1/q}$ is injective, with image $\mathcal{F} = \mathcal{K}^{1/q}[L]$. Thus, $R^{1/q} \otimes_R S \to S^{1/q}$ is injective, with image $B = R^{1/q}[S]$. We claim that $\mathcal{F} = \mathcal{L}^{1/q}$. To see this note that $[\mathcal{F} : \mathcal{K}^{1/q}] = [L : K] = [L^{1/q} : K^{1/q}]$. Hence, the fraction field of $R^[1/q][S]$ is the same as the fraction field of $S^{1/q}$, namely, $\mathcal{L}^{1/q}$. Hence, $S^{1/q}$ is contained in the integral closure of $R^{1/q}[S]$. By the Lipman-Sathaye theorem, $\mathcal{J}_{B/R^{1/q}} S^{1^q} \in B$. But if $S = T/(F_1, \dots, F_m)$ is a presentation over R, where T

is a polynomial ring over R, then $B = R^{1/q} \otimes T/(F_1, \dots, F_m)$ is a presentation over $R^{1/q}$. It follows that $\mathcal{J}_{B/R^{1/q}} = \mathcal{J}_{S/R}B$, and so $\mathcal{J}_{S/R}S^{1/q} \subseteq B$. $\square$

EC8. $\mathrm{Ass}\,(\mathrm{Hom}_R(M, N))$ is the set of associated primes of N that contain a minimal associated prime of M, i.e., that are in the support of M. To see this, let $m_1, \dots, m_h$ generate M. Then we have a map $\mathrm{Hom}_R(M, N) \to N^{\oplus h}$ such that $f \mapsto (f(m_1), \dots, f(m_n))$, and it is injective because f is 0 iff all the $f(m_i) = 0$. Hence $\mathrm{Ass}\,(\mathrm{Hom}_R(M, N)) \subseteq \mathrm{Ass}\,(N^{\oplus h} = \mathrm{Ass}\,(N)$. Also, if $Q \in \mathrm{Ass}\,(\mathrm{Hom}_R(M, N))$, the QR_Q remains amidng the associated primes when we localize at Q, and since $\mathrm{Hom}_R(M, N)_Q \cong \mathrm{Hom}_{R_Q}(M_Q, N_Q)$, this can only happen when $M_Q \neq 0$, i.e., when $Q \in \mathrm{Supp}\,(M)$. Conversely, suppose that $Q \in Supp(M)$ and $Q \in \mathrm{Ass}\,(N)$. We want to show that $Q \in \mathrm{Ass}\,(\mathrm{Hom}_R(M, N)$. Both the hypotheses and the conclusion are unaffected if we replace R, Q, M, N by R_Q, QR_Q, M_Q, N_Q, and so we may assume without loss of generality that (R, Q) is local, that $M \neq 0$, and that $Q \in Ass(N)$, so that the residue class field K injects into N as an R-module. It suffices to give a surjection $M \twoheadrightarrow K$: the composite map $M \twoheadrightarrow K \hookrightarrow N$ will have annihilator Q. But M/QM is a nonzero K-vector space by Nakayama's lemma, and so there exists a surjection of $M/QM \twoheadrightarrow K$ and hence $M \twoheadrightarrow M/QM \twoheadrightarrow K$. $\square$