Math 615, Winter 2019 **Problem Set #5**
Due: Monday, April 22

1. Let $S = K[W, X, Y, Z]$, a polynomial ring over a field K of characteristic $p > 2$, and let $R = S/(W^4 + X^4 + Y^4 + Z^4) = K[w, x, y, z]$. Let $I = (w, x, y)R$. Show that for all $p > 2$, $z^3 \in I^* - I$. For which p is it true that $(z^3)^p \in I^{[p]}$?

2. Let $R \subseteq S$ be Noetherian domains of prime characteristic $p > 0$ and suppose that there is an R-linear map $\theta : S \to R$ such that $\theta(1) \neq 0$. Prove that for every ideal I of R, $IS \cap R \subseteq I^*$.

3. Let R be Noetherian of prime characteristic $p > 0$, let $I \subseteq R$ be any ideal, and let $n > 0$ be an integer. Show that $(I^*)^{[p^n]} \subseteq (I^{[p^n]})^*$.

4. Let K be a field and let $R = K[[x^4, x^3y, xy^3, y^4]] \subseteq K[[x, y]]$, the formal power series ring. Determine the lengths of the Koszul homology modules $H_i(x^4, y^4; R)$ and determine $\chi(x^4, y^4; R)$.

5. If R is local and M is a finitely generated Rp-module, a *minima (first)l module* of syzygies of M is the kernel of a map $R^n \twoheadrightarrow M$ such that the free generators of R^n map to a minimal set of generators of M. A minimal n th module of syzygies for $n > 1$ is defined recursively as a minimal module of syzygies of a minimal $(n-1)$ th modules of syzygies.

Let $R = K[[X, Y]]/(X^2, XY) = K[[x, y]]$, where K is a field, X, Y are formal indeterminates with images x, y, and $m = (x, y)R$ is the maximal ideal. We think of K as the R-module R/m.

(a) Prove that $K \oplus m$ is a minimal first module of syzygies of m over R, and so a minimal second module of syzygies of K.

(b) Determine a minimal n th modules of syzygies of K for all $n \geq 1$, and determine the K-vector space dimensions of the modules $\mathrm{Tor}_n^R(K, K)$ for all n. Note that these are the same as the ranks of the free R-modules in a minimal resolution of K.

6. Let (R, m, K) be a local domain and let M, $A \subseteq B$ be finitely generated modules such that M_P is R_P-free for every prime P of R such that $P \neq m$. Suppose that $r \in m$ is not a zerodivisor on $M \otimes_R A$. Prove that $M \otimes_R A \to M \otimes_R B$ is injective.

Extra Credit 9. Let R be any ring. Show that an R-module M is flat if and only if $\mathrm{Tor}_1^R(R/I, M) = 0$ for every ideal $I \subseteq R$. Show that if R is Noetherian, M is flat iff $\mathrm{Tor}_1^R(R/P, M) = 0$ for every prime ideal P of R.

Extra Credit 10. Let (A, m, K) be an Artin local ring and let $x, y \in m$ be any two elements. Prove that the Koszul homology module $H_1(x, y; A)$ needs at least two minimal generators.