Math 615, Fall 2019 **Problem Set #5: Solutions**

1. Let $q = p^n = 4h + \iota_n$, where ι_n is 1 or 3. Then $(z^3)^q = (z^3)^{4h+\iota_n} = (z^4)^h z^{3\iota_n} = \pm(w^4+x^4+y^4)^{3h} z^{3\iota_n}$. When we expand, at least one of the exponents on w or x or y is $4h$. Hence, taking $c = (wxy)^3$ (not the most efficient choice), we have that $c(z^3)^q \in (w,x,y)^{[q]}$, since $4h+3 \geq q$, and so $z^3 \in (w,x,y)^*$. If $q = p$ and $\iota_1 = 1$, $z^{3p} = (w^4+x^4+y^4)^{3h} z^3$, and $1, z, z^2, z^3$ is a free basis for R over the polynomial ring $K[w,x,y]$. Every term in the expansion is in (w^p, x^p, y^p) except $\frac{(3h)!}{(h!)^3} w^{4h}x^{4h}y^{4h}z^3$, and so $z^p \notin I^{[p]}$. If $q = p$ and $\iota_1 = 3$, $z^{3p} = z^{12h+9} = (z^4)^{3h+2}z = \pm(w^4+x^4+y^4)^{3h+2}z$. When we expand, at least one of the exponents on w^4, x^4, or y^4 is at least $h+1$, and $4(h+1) > p$, so that every term is in $I^{[p]}$. Hence, $z^p \in I^{[p]}$ if and only if $p \equiv 3 \bmod 4$.

2. If $f \in IS \cap R$ then for all $q = p^n$, $f^q \in I^{[q]}S \cap R$, and when we apply θ we obtain that $f^q\theta(1) \in I^{[q]}$. Since $c = \theta(1) \neq 0$, $f \in I^*$. $\square$

3. If $f \in I^*$ we have c nonzero such that $cu^q \in I^{[q]}$ for all $q \gg 0$, where $q = p^m$. Hence, $c(u^{p^n})^q = cu^{p^n q} \in I^{[p^n q]} = (I^{[p^n]})^{[q]}$ for all $q \gg 0$, which shows that $u^{p^n} \in (I^{[p^n]})^*$. Hence, $(I^*)^{[p^n]} \subseteq (I^{[p^n]})^*$. $\square$

4. Let S be the completed Veronese subring $K[[x^4, x^3y, x^2y^2, xy^3, y^4]]$. S is a direct summand, as an S-module, of $K[[x,y]]$: the S-module complement consists of all power series involving only monomials of total degree not divisible by 4. Hence, x^4, y^4, which is a regular sequence on $K[[x,y]]$, is a regular sequence on S, and $H_i(x^4, y^4; S) = 0$ for $i \geq 1$. We have an exact sequence $0 \to R \to S \to K \to 0$ where K is the R-module R/m_R and is generated by the image of x^2y^2 (note that m_R multiplies x^2y^2 into R). $H_2(x^4, y^4; R) = 0$ since R is a domain, and so its length is 0. Since multiplication by x^4 or y^4 is 0 on K, $H_2(x^4, y^4; K) \cong K$, $H_1(x^4, y^4; K) \cong K^2$, and $H_0(x^4, y^4; K) \cong K$. The long exact sequence for Koszul homology yields $0 \to K \to H_1(x^4, y^4; R) \to 0 \to K^2 \to R/(x^4, y^4) \to S/(x^4, y^4) \to K \to 0$. Hence, $H_1(x^4, y^4; K) \cong K$ has length 1. $S/(x^4, y^4)$ is spanned over K by the monomials $1, x^3y, x^2y^2, xy^3$ and its length is 4. It follows that the length of $H_0(x^4, y^4; R)$ is $2 + 4 - 1 = 5$, and $\chi(x^4, y^4; R) = 5 - 1 + 0 = 4$.

5. One gets minimal generators for a direct sum by taking minimal generators for each summand, and it follows that a minimal first module of syzygies can be obtained as the direct sum of minimal modules of syzygies of the summands. m is minimally generated by (the images of) x and y, and $rx + sy = 0$ if and only if $r \in m$ and $s \in xR \cong K$. Thus $K \oplus m$ is a minimal first module of syzygies of m. Suppose a minimal nth module of syzygies of K has the form $K^{a_n} \oplus m^{b_n}$. By our earlier remark, $m^{a_n} \oplus (K \oplus m)^{b_n} \cong K^{b_n} \oplus m^{a_n + b_n}$ is a minimal $(n+1)$th modules of syzygies of K, i.e., $a_{n+1} = b_n$ and $b_{n+1} = a_n + b_n = b_{n-1} + b_n$ for $n \geq 1$. Since $b_0 = 0$ and $b_1 = 1$, $b_n = F_n$, the nth Fibonacci number, and a minimal nth module of syzygies $\cong K^{F_{n-1}} \oplus m^{F_n}$. The least number of generators of the $n\,th$ module of syzygies is $F_{n-1} + 2F_n = (F_{n-1} + F_n) + F_n = F_{n+1} + F_n = F_{n+2}$, and this is the same as the rank of nth free module in a minimal free resolution. Thus $\mathrm{Tor}_n^R(K,K) \cong K^{F_{n+2}}$.

6. Let $C = B/A$. The hypothesis implies that the finitely generated R-module $\mathrm{Tor}_1^R(M,C)_P \cong \mathrm{Tor}_1^{R_P}(M_P, C_P) = 0$, and so $\mathrm{Tor}_1^R(M,C)$ is supported only at m and has finite length. In the long exact sequence for Tor coming from $0 \to A \to B \to C \to 0$, consider

$\mathrm{Tor}_1^R(M, C) \to M \otimes_R A \to M \otimes_R B$. The image of $\mathrm{Tor}_1^R(M, C)$ wiill have finite length and so be killed by a power of r. Since r is not a zerodivisor on $M \otimes_R A$, the image must be zero, which proves the required injectivity. (It is not needed that R is a domain.) $\square$

EC9. If M is flat, $\mathrm{Tor}_i^R(N, M) = 0$ for all N and $i \geq 1$ (a projective resolution of N remains acyclic when one tensors with M). From the long exact sequence for Tor it follows that M is flat if $\mathrm{Tor}_1^R(N, M) = 0$ for all N, so this condition is necessary and sufficient. Since N is a directed union of finitely generated modules and Tor commutes with direct limit, it suffices if $\mathrm{Tor}_1^R(N, M) = 0$ when N is finitely generated. In this case, N has a filtration by cyclic (in the Noetherian case, prime cyclic) modules. We now prove that if $\mathrm{Tor}^R(N, M)$ vanishes when N is cyclic (resp., prime cyclic) then it vanishes for all finitely generated modules N. This follows by induction on the length n of the filtration: the case $n = 1$ is given, and, for the inductive step, there is a short exact sequence $0 \to R/I \to N \to N' \to 0$ (where I is prime in the Noetherian case), and N' has a filtration whose length is shorter than the filtration of N. We may apply the long exact sequence for Tor (we only need the three Tor_1^R terms) to conclude the proof. $\square$

EC10. Note that there is a nonzero element of A killed by m: if n is maximum such that $m^n \neq 0$, any element of m^n has this property. (If A is a field, $m = (0)$, $n = 0$, and 1 is such an element.) Hence $\mathrm{Ann}_A(x, y) = H_2(x, y; A) \neq 0$. Since the sequence x, y has length $2 > 0 = \dim(A)$, $\chi(x, y; A) = 0$, and so $\ell\big(H_1(x, y; A)\big) = \ell(A/(x, y)A) + \ell\big(\mathrm{Ann}_A(x, y)\big) > \ell\big(A/(x, y)A\big)$. If $H_1(x, y; A)$ has one or 0 generators, then, since it is killed by (x, y), it would have length at most $\ell\big(A/(x, y)A\big)$, a contradiction. Hence, $H_1(x, y; A)$ must have two or more generators. $\square$