Math 615, Winter 2020 **Extra Credit Problem Solutions**

EC1. Let the homogeneous generators be $F_1, \ldots, F_n$ with respective degrees $d_1, \ldots, d_n$ and let L be the least common multiple of $d_1, \ldots, d_n$. Then every monomial μ in the F_i of degree $D \geq nL$ is the product of a monomial of degree L and one of degree $D - L$: the fact that $\mu = F^{a_1} \cdots F_k^{a_k}$ has degree D implies that $\sum_{i=1}^n d_i a_i \geq nL$, and so at least one $d_i a_i \geq L$. Then we can choose $b \leq a_i$ such that $d_i b = L$, and F_i^b has degree L and is a factor of μ. If μ has degree nLh for $h > 1$ we can iterate this $n(h-1)$ times to write μ as a product of the $n(h-1)$ forms of degree L and one of degree nL. The former term can be written as a product of $h-1$ forms of degree nL by grouping. Thus, every monomial of degree nLh is a product of h monomials of degree nL, and we may take $k = nL$. $\square$

EC2. The degree n homogeneous component of $T := K[y_1, \ldots, y_r] \# K[z_1, \ldots, z_s]$ consists of the products of a degree n monomial in the y_i and a degree n monomial in the z_j. If we write the variables in each of the monomials as an ordered sequence then one monomial is $y_{i_1} \cdots z_{i_n}$ with $1 \leq i_1 \leq \cdots \leq ik_n \leq r$ and the other is $z_{j_1} \cdots z_{j_n}$ with $1 \leq j_1 \leq \cdots \leq j_n \leq s$. Their product is is the image of $\mu := x_{i_1,j_1} \cdots x_{i_n,j_n}$ under the map from $K[X] \twoheadrightarrow T$. If we partially order the indeterminates so that $x_{hi} \leq x_{jk}$ if $h \leq j$ and $i \leq k$, then the monomials μ as above correspond those of degree n in the x_{ij} such that the variables occurring are linearly ordered. We take these to be the family $\mathcal{B}$. Note that $1 \in \mathcal{B}$. We claim that every monomial of degree n in the x_{ij} is congruent to a monomial in $\mathcal{B}$ modulo I. To see this, write the monomial as $x_{i_1,j_1} \cdots x_{i_n,j_n}$ with $i_1 \leq \cdots \leq i_n$. It suffices to show that this is monomial is congruent to $x_{i_1}, j_{\pi(1)} \cdots x_{i_n}, j_{\pi(n)}$ modulo I for every permutation π of $1, \ldots, n$. Since every permutation is a product of transpositions, it suffices to show this when π is a transposition. But this follows from the fact that for all u and v, $x_{i_u,j_u} x_{i_v,j_v} \equiv x_{i_u,j_v} x_{i_v,j_u}$ modulo the ideal of 2×2 minors of the matrix. We now claim that the kernel of $K[X] \twoheadrightarrow T$ must be exactly I. We know that every element iof $K[X]$ modulo I can be written as an element of K-span of $\mathcal{B}$. Hence, an element of the kernel outside I that mapped to 0 in T would be a non-trivial K-linear combination of elements of $\mathcal{B}$. But such an element cannot map to 0 in T, because $\mathcal{B}$ maps bijectively onto a K-basis for T. $\square$

EC3. Since Q is the cokernel of α, it is obtained from T^n by forming the quotient by the column space of α. Since Q is a (T/fT)-module, the image of every fe_j in Q is 0, and so is a linear combination of the columns of α, as stated. The fact that $fI_n = \alpha\beta$ is equivalent to the statement that the j th columns of fI_n, which is fe_j written as column, is α times the j th column of β, which is v_j. Since f is assumed to be a nonzerodivisor in m, if we think of $T \subseteq T_f$ then $fI = \alpha\beta$ implies that $f^{-1}\alpha$ and β are inverses. Hence, their product is the identity in either order, and $\beta f^{-1}\alpha = I$ as well. Hence,$f^{-1}\beta\alpha = I$, and multiplying by f gives the result. Finally, we shall show that if $\alpha\beta = \beta\alpha = fI$, then over T/fT the image of α and β is the kernel of the other. It is clear that modulo fT, $\alpha\beta = \beta\alpha = fI \equiv 0$. By symmetry, it well suffice to prove that if $\alpha u \equiv 0$ modulo fT, then u is in the image of β mod fT. But the fact that $\alpha u = 0$ in fT means that $\alpha u = fw$ over T, for a suitable column vector w in T^n. But then $\alpha u = \alpha\beta w$ and $\alpha(u - \beta w) = 0$ over T and over T_f. But over T_f, the matrix α is invertible, and so $u = \beta w$. $\square$

EC4. R is not Noetherian. There are many arguments that prove this — we give only one. If $n \geq 1$, the product of any n consecutive positive integers, say $k, k+1, \ldots, k+n-1$, is divisible by $n!$ (the quotient is $\binom{k+n-1}{n}$). Hence, the same is true for the product of any n consecutive negative integers. Thus, if $n \geq 1$ and $f_n = x(x-1)\cdots(x-n+1)/n!$, then f_n maps $\mathbb{Z}$ to $\mathbb{Z}$ (if a factor in the numerator is 0, the value is 0). Let $I_n = (f_1, \ldots, f_n)R$ and $I = \bigcup_{n=1}^{\infty} I_n$. If I is finitely generated, then $I = I_n$ for some $n \gg 0$. Thus, it suffices to show that if p is any odd prime, $f_p \notin I_{p-1}$. Suppose $(*)$ $\quad f_p = \sum_{j=1}^{p-1} g_j f_j$ with all $g_j \in R$. Evaluate both sides at $2p$ to get $(**)$ $\quad \binom{2p}{p} = \sum_{j=1}^{p-1} g_j(p)\binom{2p}{j}$. p occurs as a factor twice in $(2p)!$ (in $2p$ and in p) and twice in $(p!)^2$. Hence, p does not divide $\binom{2p}{p}$. Every $g_j(p) \in \mathbb{Z}$, since $g_j \in R$. But $p|\binom{2p}{j}$ for $1 \leq p-1$, since it occurs as a factor in the numerator $(2p)(2p-1)\cdots(2p-j-1)$ while it does not occur as a factor in the denominator $j!$. This is a contradiction, since $(**)$ then implies $p|\binom{2p}{p}$. $\square$

EC5. To show the Cohen-Macaulay property when R is a polynomial ring, first note that by **EC2.**, this Segre product is isomorphic with the quotient of a polynomial ring in $2d$ variables by the ideal generated by the 2×2 minors of $2 \times d$ matrix, and that the quotient by the ideal J generated by the specified system of parameters consisting of linear forms has length $d+1$, which is equal to the multiplicity of R computed from the Hilbert polynomial. This plies that R is Cohen-Macaulay. In fact, R is a finitely generated module over the polynomial ring A in the system of parameters consisting of linear forms. By the homogeneous form of Nakayama's lemma, the least number of generators of R as an A-module is the length of $R/m_A R$, and by the remark above this is $s+1$. The multiplicity of R as an A-module is the same as the torsion-free rank h of R over A: we have $A^h \subseteq R \to N \to 0$, where N is A-torsion and has Krull dimension smaller than that of A. Computed asymptotically, The multiplicity of R over A with respect to m_A is the same as its multiplicity with respect to the ideal $m_A R = J$ generated by the linear system of parameters, and since this ideal is contained in m_R, the multiplicity of R (with respect to m_R), which we know to be $d+1$, is at most the multiplicity of R with respect J, which is h. Thus, $s+1 \leq h$. Since R is generated by $s+1$ elements as an A-module, we also have $h \leq s+1$. Thus, $h = s+1$ and this means that R is free over A and therefore Cohen-Macaulay. Of course, a completely different proof of a much more general result is given in the installment of the Lecture Notes from the course.

To show that the Segre product is *not* Cohen-Macaulay in the situation where $R = K[X_1, \ldots, X_n]/(F) = K[x_1, \ldots, x_n]$ with F as described, note that R is a module-finite extension of $B := K[x_1, \ldots, x_{n-1}]\#_K K[y,z]$, so that, with $d = n-1$, we have that $x_1 z, x_1 y - x_2 z, \ldots, x_i y - x_{i+1} z, \ldots, x_{n-2} y - x_{n-1} z, x_{n-1} y$ is a homogeneous system of parameters for B and, hence, for R. Thus, it suffices to show that these parameters are not a regular sequence in R.

But $(x_n y)^{n-1}(x_1 z) = \big((x_n y)^{n-2}(x_n z)\big)(x_1 y - x_2 z) + \big((x_n y)^{n-2}(x_n z)\big)(x_2 z)$
$= \big((x_n y)^{-n2}(x_n z)\big)(x_1 y - x_2 z) + \big((x_n y)^{n-3}(x_n z)^2\big)(x_2 y - x_3 z) + \big((x_n y)^{n-3}(x_n z)^2\big)\big(x_3 z\big)$.
We can continue in this way, each time making a substitution for the rightmost term. If we write $L_0 = x_1 z$, $L_i = x_i y - x_{i+1} z$, $1 \leq i \leq n-1$, and $L_{n-1} = x_{n-1} y$, we eventually get

the expression

$$(x_n y)^{n-1} L_0 = \sum_{i=1}^{n-1} (x_n y)^{n-1-i} (x_n z)^i L_i.$$

We can then see that the elements L_0, $.L_1, \ldots, L_{n-1}$ are not a regular sequence (which would be permutable in the regular case) because $(x_n y)^{n-1} \notin (L_1, \ldots, L_{n-1})$. To see this, note that there is an R-homomorphism of $R[y,z] \to R$ such that $y \mapsto 1$ and $z \mapsto 0$. This restricts to a homomorphism $\theta : R\#_K[y,z] \to R$, and if we had $(x_n y)^{n-1} \in (L_1, \ldots, L_{n-1})$, applying the homomorphism would yield $\theta(x_n y)^{n-1} = x_n^{n-1} \in (x_1, \ldots, x_{n-1})R$ since $\theta(L_i) = x_i$ for $1 \leq i \leq n-1$. This is false, because the degree of f is $r \geq n$, and so $R/(x_1, \ldots, x_{n-1} \cong K[x_n]/(x_n^r)$, with $r \geq n$, and the image of x_n^{n-1} is not 0.

EC6. Flatness is equivalent to the assertion that if $A \subseteq B$ then $A \otimes M \to B \otimes M$ is injective. Since M is a direct union of submodules finitely generated over the image of A, tensor commutes with direct limit, and a direct limit of injections is an injection, it suffices to prove this when B is finitely generated over A. We may then insert a chain of modules $B = B_0 \subseteq B_1 \subseteq \cdots \subseteq B_s = B$ such that B_{i+1} is generated by one element over B_i, $0 \leq i \leq s$. Therefore, flatness is equivalent to the assertion that if $A \subseteq B$ and B/A is cyclic, say $B/A \cong R/I$, then $A \otimes M \to \otimes M$ is injective. By the long exact sequence fo Tor, it suffices if $\mathrm{Tor}_1(R/I, M) = 0$ for every ideal I of R. Since R/I is the direct limit of R/I_0 as I_0 runs through the finitely generated subideals of I and Tor commutes with direct limit, M is flat iff $\mathrm{Tor}_1(R/I, M) = 0$ for every *finitely generated* ideal $I = (f_1, \ldots, f_n)$ of R. A free resoluton of R/I begins $\cdots \to G \xrightarrow{\beta} R^n \xrightarrow{\alpha} R \to R/I \to 0$ where α is the map taking the i th free generator of R^n to $f_i \in R$ and the generators of G map under β to generators of the relations on $f_1, \ldots, f_n$ (where a relation is an n-tuple $(r_1, \ldots, r_n) \in R^n$ such that $\sum_{i=1}^n r_i f_i = 0$): we can even assume that the generators of G map to all relations over R on $f_1, \ldots, f_n$ The vanishing of $\mathrm{Tor}_1(R/I, \ )$ is then equivalent to the exactness of $\cdots \to G \otimes M \xrightarrow{\beta_M} M^n \xrightarrow{\alpha_M} M$ at the middle spot, where the subscript $_M$ on the map indicates the tensor product with the identity map on M. Here, α_M sends $(u_1, \ldots, u_n) \mapsto \sum_{i=1}^n f_i u_i$. Exactness at the middle spot therefore says precisely that every relation on the f_i with coefficients in M is an M-linear combination of the relations on the f_i with coefficients in R. $\square$

EC7. By the Theorem on p. 88 of the Lecture Notes, we have $G_0(R/yR) \to G_0(R) \to G_0(R_y) \to 0$ is exact. Note that $R/(yR) \cong K[X,U,V]/(XU)$, and since the quotients by the minimal primes generated by the images of X and U are polynomial, $G_0(R/xR)$ is generated by $[R/(x,y)]$ and $[R/(u,y)]$. Since $R_y \cong K[X,U,Y][1/Y]$ (we can use the defining equation to solve for V), the Grothendieck group of R_y is $\mathbb{Z}\cdot[R_y]$, which is the image of $\mathbb{Z}\cdot[R]$ in in $G_0(R)$. It follows that $G_0(R)$ is generated by the classes $[R/P] = [R/(x,y)]$, $R/(u,y)]$, and $[R]$. In $G_0(R)$ we have a short exact sequence $0 \to R/xyR \to R/(x,y) \oplus R/(u,y) \to R/(x,u,y) \to 0$. In $G_0(R)$, $[R/xyR] = [R] - [xyR] = [R] - [R] = 0$, and $R(x,y,u) = [R(x,y) - [uR/(x,y)] = 0$, so that $R/(u,y) = -[R/P]$ and is not needed as a generator. Hence, $G_0(R)$ is generated by the $[R]$ and $[R/P]$, and the reduced Grothendieck group $\overline{G}_0(R)$ is generated by the image of $[R/P]$. It follows that if the divisor class group of R has infinitely many elements, it must be $\mathbb{Z}$, since it is cyclic, and then it follows

that $[R/P$ generates $\mathbb{Z} \cong \overline{G}_0(R)$, and this implies $G_{\bullet 0}(R)$ is $\mathbb{Z} \oplus \mathbb{Z}$. It will therefore suffice to show that the pure height one ideals $P^{(n)}$ are distinct as R-modules. Think of $R \cong K[sw, sz, tw, tz] \subseteq T := K[s, t, w, z]$ where x, y, u, v correspond to sw, sz, tz, tw respectively. Then $P = sT \cap R$, and $P_n = s^n T \cap R$ is generated minimally by $s^n w^i z^{n-i}$, $0 \leq i \leq n$ and is primary to P. It follows easily that $P_n = P^{(n)}$, and needs $n+1$ minimal generators. Thus, the $P^{(n)}$ are all distinct as modules. $\square$

EC8. Note that $0 \to A \to B \xrightarrow{\beta} C \to 0$ is split iff $\theta : \mathrm{Hom}(C, B) \to \mathrm{Hom}(C, C)$ is onto: this is clearly the case if $B = A \oplus C$, while if θ is onto a splitting is given by an element of $\mathrm{Hom}(C, B)$ that maps to the identity map on C. In general, one has

$$(*) \quad 0 \to \mathrm{Hom}(C, A) \to \mathrm{Hom}(C, B) \xrightarrow{\theta} \mathrm{Hom}(C, C) \to Q \to 0$$

is exact with some cokernel Q. The issue of whether Q is 0 is local on the prime ideals of the base ring R, so we may assume that $(R, \mathfrak{m})$ is local. If the modules have finite length, and $B \cong A \oplus C$ (possibly in a different short exact sequence), still have $\mathrm{Hom}(C, B) \cong \mathrm{Hom}(C, A) \oplus \mathrm{Hom}(C, C)$, from which the alternating sum of the lengths of the leftmost three nontrivial modules in $(*)$ is 0. It follows that the length of Q is 0, and so $Q = 0$, as required.

Now suppose $(R, \mathfrak{m})$ is local but that we do not assume finite length. f Since flat base change to $\widehat{R}$ commutes with Hom, proving the surjectivity of β, i.e., proving that one has a splitting, reduces to the case where R is complete local. Note that for every n, we have an exact sequence $0 \to V_n \to A/m^n A \to B/m^n B \xrightarrow{\overline{\theta}} C/m^n C \to 0$, and that $B/m^n B \cong A/m^n A \oplus B/m^n B$ in some way. The alternating sum of the lengths of the three rightmost nontrivial modules is 0, and so the length of V_n is 0 and we actually have a short exact sequence. By the finite length result, we have that $0 \to A/m^n A \to B/m^n B \to C/m^n C \to 0$ is split for all n, so that $\theta_n : \mathrm{Hom}(C/m^n C, B/m^n B) \to \mathrm{Hom}(C/m^n C, C/m^n C)$ is split, this implies $\mathrm{id}_{C/m^n C}$ is the composition of a map $f_n : C/m^n C \to B/m^n B$ with the map $\beta_n : B/m^n B \to C/m^n C$ induced by β. The set of splittings of $B/m^n C \to B/m^n C$ is the coset $Y_n = f_n + W_n$, where W_n is the kernel of the map $\mathrm{Hom}(C/m^n C, B/m^n B) \to \mathrm{Hom}(C/m^n C, C/m^n C)$ that sends $f \mapsto \beta_n \circ f$. To complete the proof, we note that these cosets form an inverse limit system, and that an element of the inverse limit will induce a map $C \to B$ that gives a splitting. Thus, we must show the inverse limit of the Y_n is nonempty. We define the length of a coset a finite length module to be the length of the finite length module. Note that the images of Y_j in Y_n for $j \geq n$ are a nonincreasing sequence of cosets, and so their lengths are nonincreasing and are eventually constant. This implies that the image of Y_j in Y_n for all $j \gg n$ is the same. Call the stable image W_n. Then the W_n form an inverse limit of cosets and surjective maps, and we can recursively construct an element of the inverse limit, which is also in the inverse limit of the Y_n. $\square$

EC9. We may assume p exceeds all the a_i. Let $a := a_n$. From the fact in brackets, $a x_n^{a-1}$ is a test element, and since a is an invertible scalar, we may use $c = x^{a-1}$ in all tight closure tests. The issue is then whether $c(x_n^{a_n - 1})^q \in I^{[q]}$, i.e., whether

$$(*) \quad x_n^{(a-1)(q+1)} \in (x_1^{a_1 q}, \ldots, x_{n-1}^{a_{n-1} q}) R$$

for all $q = p^e \gg 0$. Let $T = K[x_1, \ldots, x_{n-1}]$. Let $J = (x_1^{a_1}, \ldots, x_{n-1}^{a_{n-1}})T \subseteq T$. Then R is the free module of rank a over T with free basis $1, x_n, x_n^2, \ldots, x_n^{a-1}$. Let $g = x_1^{a_1} + \cdots + x_{n-1}^{a_{n-1}} \in T$, so that $x_n^a = -g$ in R. Write $q+1 = va - r$, where $0 \leq r \leq a-1$. Then, in R, we have $x_n^{(a-1)(q+1)} = x_n^{a(q+1)-va+r} = (x_n^a)^{q+1-v} x_n^r = g^{q+1-v} x_n^r$, and since $g \in T$ the issue of whether this is in $(x_1^{a_1 q}, \ldots, x_{n-1}^{a_{n-1} q})(T + Tx_1 + \cdots + Tx_n^r + \cdots + Tx_n^{a-1})$ is whether $g^{q+1-v} \in (x_1^{a_1 q}, \ldots, x_{n-1}^{a_{n-1} q})T = J^{[q]} \subseteq T$. The issue is therefore with every term in the multinomial expansion g is in J^q. The terms don't cancel, but the multinomial coefficient may be zero. Therefore, we can complete the proof by finding nonnegative integers $s_1, \ldots, s_{n-1}$ such that (1) $\sum_i s_i = q + 1 - v$, (2) the multinomial coefficient $\binom{s}{s_1, \ldots, s_{n-1}}$ is not 0 modulo p, and (3) for all i, $a_i s_i$ (the exponent on x_i in the term of the multinomial expansion) is $< a_i q$.

We shall achieve this while taking all of the s_i to have the form $t_i p^{e-1}$, where $0 \leq t_i \leq p-1$ and, moreover, $t := t_1 + \cdots + t_{n-1} \leq p-1$. Let $t = \lceil (q-v+1)/p^{e-1} \rceil$, and let $w_i = \lfloor (p/a_i) \rfloor < p/a_i$. First observe that t is at most one more than $(q-v+1)/p^{e-1} = p + (1/p^{e-1}) - v/p^{e-1}$ where $v = (q+1+r)/a$ as in the preceding paragraph, so that $v/p^{e-1} = (p/a) + \big((1+r)/p^{e-1}a\big)$. Thus

$$t \leq 1 + p + (1/p^{e-1}) - v/p^{e-1} = 1 + p + (1/p^{e-1}) - \Big((p/a) + \big((1+r)/p^{e-1}a\big)\Big) =$$

$$1 + p - p/a + 1/p^{e-1} - (1+r)/p^{e-1}a \leq p - p/a + 1 + 1/p^{e-1} \leq p - (p/a - 1 - 1) < p$$

provided $p > 2a$. Note that $a_i(w_i p^{e-1}) < a_i(p/a_i)p^{e-1} = q$, and that $wp^{e-1} \geq q - v + 2$. We do not have that $t = w_1 + \cdots + w_{n-1}$, but we have, in fact, that $w_1 + \cdots + w_{n-1} \geq t$. To see this, note that $t \leq \big((q-v+1)/p^{e-1}\big) + 1 = p + 1 - (v-1)/p^{e-1}$ and that $w_i \geq (p/a_i) - 1$. Hence, it will suffice if $(\sum_{i=1}^{n-1} p/a_i) - (n-1) \geq p + 1 - (v-1)/p^{e-1}$. We transpose the $-(n-1)$ term to the other side of the equation, substitute $v = (q+1+r)/a$, and divide by p to see that it will suffice if we have the inequality $\sum_{i=1}^{n-1} 1/a_i \geq \big((n-1)/p)\big) + 1 + (1/p) - (q+1+r-a)/aq = 1 + (n/p) - (q+1+r-a)/aq$. If $p \geq a$, the third term on the right is ≤ 0, and it evidently will then suffice if $\sum_{i=1}^{n-1} 1/a_i \geq 1 + (n/p)$. Since $\sum_{i=1}^{n-1} 1/a_i > 1$, this is clearly true for all $p \gg 0$ — note that all the a_i and n are fixed. Thus, for $p \gg 0$ we have that the $w_1 + \cdots + w_{n-1} \geq t$. Therefore, by decreasing the w_i suitably, we can choose nonnegative integers t_i such that $t_i \leq w_i$, $1 \leq i \leq n-1$, and $\sum_{i=1}^{n-1} t_i = t$.

We now take $s_i = t_i p^{e-1}$ and $s = tp^{e-1}$. The argument will be complete if we can show that the multinomial coefficient $\binom{s}{s_1, \ldots, s_{n-1}}$ does not vanish mod p. This is true because it is equal mod p to the multinomial coefficient $\binom{t}{t_1, \ldots, t_{n-1}}$. One can see this by expanding $(z_1 + \cdots + z_{n-1})^{tp^e}$ in two ways: on the one hand, the coefficient of $z_1^{t_1 p^{e-1}} \cdots z_{n-1}^{t_{n-1} p^{e-1}}$ is $\binom{s}{s_1, \ldots, s_{n-1}}$. On the other hand, we can think of the expansion as $((z_1 + \cdots + z_{n-1})^t)^{p^{e-1}}$, which gives the coefficient of the required term as $\binom{t}{t_1, \ldots, t_{n-1}}^{p^{e-1}}$, which is the same as $\binom{t}{t_1, \ldots, t_{n-1}}$. But now we can see that this does not vanish, simply because $t < p$. $\square$

EC10. Let $R = K[x^2, xy, y^2] \subseteq K[x, y]$. Let $I = (x^2)$, $J = (xy)$. Then $I :_R J = (x^2, xy)R$ and $(I :_R J)^{[p]} = (x^{2p}, x^p y^p)R$, while $I^{[p]} :_R J^{[p]} = (x^{2p}) :_R (x^p y^p)$ consists of all even degree multiples of x^p. If $p = 2$, x^2 is in this ideal, while if p is odd, x^{p+1} and $x^p y$ are in this ideal.