Math 615, Winter 2020: Lecture of Monday, March 30

In the first part of this lecture we use the optional results on convexity over the rational numbers from the previous lecture to finish the proof of a missing Lemma, and with that we will have completed the proof that normal rings generated by monomials are Cohen-Macaulay.

We then begin our introduction to tight closure theory. In fact, the basic idea of tight closure is in the proof of our theorem on colon-capturing in positive characteristic.

In this paragraph, all rings are Noetherian, of prime characteristic $p > 0$. For simplicity, think of the case where R is a domain, and consider an ideal I. Suppose that $u \in R$. The idea is that if there is a fixed nonzero element $c \in R$ such that $cu^{p^e} \in I^{[p^e]}$ for all $e \gg 0$ (where $I^{[p^e]} = (f^{p^e} : f \in I)R$) then u is "almost" in I in some sense. It turns out that when R is regular, this condition implies that u *is* in I. When R is not regular, this condition becomes a definition: u is said to be in the *tight closure* of I. The word "tight" is used because this closure is a "tight fit" for the ideal, i.e., it is small compared to other closures. We will extend this notion to a closure operation on submodules of finitely generated modules.

It turns out that many rings besides regular rings have the property that every ideal is tightly closed. For example, the rings of the form $K[X]/I_t(X)$, where X is a matrix of indeterminates and $I_t(X)$ is the ideal generated by the $t \times t$ minors of X have this property. The same is true for the ring generated by the $r \times r$ minors of an $r \times s$ matrix of indeterminates, where $1 \le r \le s$, and for normal subrings of the Laurent polynomials over a field that are generated by finitely many monomials. A number of theorems that hold for regular rings hold much more generally if one changes the conclusion, for example, so that instead of saying that an element satisfying certain condition is in an ideal, one says instead that it is in the tight closure of the ideal.

Rings in which every ideal is tightly closed are called *weakly* F-*regular*. If the same condition holds for all localizations, the ring is called F-*regular*. Major results include the fact that weakly F-regular rings are both Cohen-Macaulay and normal.

We shall also indicate how the theory may be extended to Noetherian rings containing a field of characteristic 0.

We now return to the final step in the proof of the Cohen-Macaulay property for normal rings generated by monomials.

In the previous lecture we established the results that we need about convex geometry over the rational numbers, and we are now ready to prove the Lemma on p. 150 of the Lecture Notes of March 25, which will also complete the proof that normal subrings of $K[x_1, 1/x_1, \ldots, x_n, 1/x_n]$ generated by finitely many monomials are Cohen-Macaulay.

Proof of the Lemma on embedding normal subsemigroups as full subsemigroups of $\mathbb{N}^s$. Let $H \subseteq \mathbb{Z}^n$ be a finitely generated normal subsemigroup that does not contain the additive inverse of any of its nonzero elements. We want to show that H can be embedded as a full subsemigroup in $\mathbb{N}^s$ for some s. First note that $H - H$ is a free abelian group, and so we may replace $\mathbb{Z}^n$ by $H - H$. Henceforth, we assume that $H - H = \mathbb{Z}^n$. This does not affect the condition that H be normal. Second, let $C = \mathbb{Q}^+ H$ be the $\mathbb{Q}^+$-subsemigroup generated by H. It is generated over $\mathbb{Q}^+$ by the generators of H, and so is finitely generated as a $\mathbb{Q}^+$-subsemigroup of $\mathbb{Q}^n$. It contains no line, for if we had β and $-\beta$ both in $\mathbb{Q}^+ H$, we could choose a positive integer N such that $N\alpha, -N\alpha \in H$, a contradiction.

Let $\alpha_1, \ldots, \alpha_h$ be nonzero generators of H, and, hence, of C. Let $V = \mathbb{Q}^n$ and $V^* = \mathrm{Hom}_{\mathbb{Q}}(V, \mathbb{Q})$. Let $C' \subseteq V^*$ be the set of all linear functionals in V^* that are nonnegative on C. Since all elements of C are nonnegative rational linear combinations of $\alpha_1, \ldots, \alpha_h$,

$$C' = G_1 \cap \cdots \cap G_h,$$

where

$$G_j = \{L \in V^* : L(\alpha_j) \geq 0\}$$

for $1 \leq j \leq h$. We may think of α_j as an element of $(V^*)^* \cong V$. Then every G_j is a half-space in V^*, and so C' is a fintely generated $\mathbb{Q}^+$-subsemigroup in V^*. Choose $L_1, \ldots, L_s \in V^*$ that generate C' over $\mathbb{Q}^+$. Each $L_i(\alpha_j)$ is nonnegative rational number. We may therefore replace L_i by a multiple by a suitable positive integer, and so assume that for all i, j, the value of $L_i(\alpha_j)$ is in $\mathbb{N}$. Since every element of H is a linear combination of the α_j with coefficients in $\mathbb{N}$, it follows that all values of every L_i on H are in $\mathbb{N}$. We therefore have a map

$$\Phi = (L_1, \ldots, L_s) : H \to \mathbb{N}^s$$

where

$$\alpha \mapsto \big(L_1(\alpha), \ldots, L_s(\alpha)\big).$$

To complete the proof, we shall show that this map is one-to-one and that its image in $\mathbb{N}^s$ is a full subsemigroup of $\mathbb{N}^s$. First, suppose that $\alpha, \beta \in H$ are distinct. Then $\alpha - \beta$ is nonzero, and so either $\alpha - \beta \notin H$ or $\beta - \alpha \notin H$. Suppose, say, that $\alpha - \beta \notin H$. The $\alpha - \beta \notin C$ as well: otherwise, $k(\alpha - \beta) \in H$ for some integer $k > 0$, and, since H is normal, we then have $\alpha - \beta \in H$, a contradiction. Hence, there is a linear functional nonnegative on C and negaqtive on $\alpha - \beta$. This linear functional is in C' and so is a nonnegative rational linear combination of the L_i. It follows that some L_i is negative on $\alpha - \beta$. But then $L_i(\alpha) \neq L_i(\beta)$. Thus, Φ is injective.

Finally, we need to show that the image of H under Φ is a full subsemigroup of $\mathbb{N}^s$. Suppose that $\Phi(\alpha) - \Phi(\alpha') \in \mathbb{N}^s$. We want to show that $\alpha - \alpha' \in H$. But $\Phi(\alpha - \alpha') \in \mathbb{N}^s$, and so $L_i(\alpha - \alpha') \geq 0$ for al i. If $\alpha - \alpha' \notin C$, we know that there is a linear functional L that is nonnegative on C and negative on $\alpha - \alpha'$. But then $L \in C'$, and this is impossible because every L_i is nonnegative on $\alpha - \alpha'$. Thus, $\alpha - \alpha' \in C$. But then for some positive integer k, we have that $k(\alpha - \alpha') \in H$, and so $\alpha - \alpha' \in H$, since H is normal. $\square$

Tight closure

We have shown in a graded instance that a direct summand of a polynomial ring is Cohen-Macaulay, and we have applied that result to show that finitely generated integrally closed rings generated by monomials are also Cohen-Macaulay.

The idea of the proof can be used to establish the result in much greater generality. In fact, it is known that if R is a Noetherian regular ring containing a field and $A \subseteq R$ is a direct summand of R as A-modules, then A is Cohen-Macaulay. Recently, perfectoid methods have been used to extend this result to the case where R is a regular ring that does not necessarily contain a field, like a polynomial or formal power series ring in finitely many variables over $\mathbb{Z}$ or over a Noetherian discrete valuation domain, e.g., over the p-adic integers. But the perfectoid proof rests on positive characteristic results.

The tool that one needs to establish this result in characteristic $p > 0$ is called *tight closure theory.* A similar theory, defined by reduction to positive characteristic, exists for Noetherian rings containing the rationals. Whether there exists a comparable theory for rings that need not contain a field is a very important open question, and new ideas from perfectoid geometry may provide a solution.

We are going to develop part of the theory in positive characteristic, and explain how the theory is extended to rings that contain $\mathbb{Q}$ without giving full details. We shall also explain why having such a theory would solve many open problems in mixed characteristic.

We begin by defining tight closure for ideals in Noetherian rings of positive prime characteristic p, and discussing some of its good properties. The notion was introduced implicitly in the Theorem on colon-capturing, which is the second Theorem on p. 125 of the Lecture Notes of March 11, but the explicit definition was not made at that point.

Definition: tight closure. Let R be a Noetherian ring of prime characteristic $p > 0$, let I be an ideal of R, and let $f \in R$. We say that f is in the *tight closure* of I if there exists an element $c \in R$, not in any minimal prime of R, such that for all $e \gg 0$, $cf^{p^e} \in I^{[p^e]}$. The set of elements in the tight closure of I is called the *tight closure* of I, and is denoted I^*.

In the earlier Theorem on colon-capturing, R was a domain. Notice that when R is a domain, the condition that c not be in any minimal prime of R is simply the condition that c not be 0. We note some elementary properties of the tight closure operation. Until further notice, R is a Noetherian ring of prime characteristic $p > 0$.

(1) *I^* is an ideal of R, and $I \subseteq I^*$. If $I \subseteq J \subseteq R$ are ideals, then $I^* \subseteq J^*$.*

As we did earlier in this context, we use q to stand for p^e. If $cf^q \in I^{[q]}$ for all $q \gg 0$, then $c(rf)^q \in I^{[q]}$ for all $q \gg 0$. If also $c'g^q \in I^{[q]}$ for all $q \gg 0$, then $(cc')(f+g)^q = c'cf^q + cc'g^q \in I^{[q]}$ for all $q \gg 0$. If $f \in I$ then $1 \cdot f^q \in I^{[q]}$ for all q, which shows that $I \subseteq I^*$. The fact that $I \subseteq J \Rightarrow I^* \subseteq J^*$ is obvious from the definition. □

We shall use the notation R° for the set of elements of R not in any minimal prime of R. The element c used in checking whether a given element of $u \in R$ is in I^* is allowed to depend on u. However, there is a single element $c \in R^\circ$ that can be used for all elements of I^*: that is, if $u \in I^*$, then $cu^q \in I^{[q]}$ for all $q \gg 0$. The point is that I^* is finitely generated: suppose that $u_1, \ldots, u_h$ are generators. Let $c_j \in R^\circ$ be such that $c_j u_j^q \in I^{[q]}$ for all $q \gg 0$, $1 \leq j \leq h$. Let $c = c_1 \cdots c_h$. Then since every $u \in I^*$ is an R-linear combination of $u_1, \ldots, u_h$, we have that $cu^q \in I^{[q]}$ for all $q \gg 0$. This implies that $c(I^*)^{[q]} \subseteq I^{[q]}$ for all $q \gg 0$.

One can use this to see that $(I^*)^* = I^*$. For suppose that u is such that $c'u^q \in (I^*)^{[q]}$ for all $q \gg 0$. Then $(cc')u^q = c(c'u^q) \in c(I^*)^{[q]} \subseteq I^{[q]}$ for all $q \gg 0$, and so $u \in I^*$. We state this formally:

(2) *If I is any ideal of R, $(I^*)^* = I^*$.*

We note that if R is a domain or if I is not contained in any minimal prime of R, then $u \in I^*$ iff there exists $c \in R^\circ$ such that $cu^q \in I^{[q]}$ for all q. In the second case we can choose $c' \in I - R^\circ$. If $cu^q \in I^{[q]}$ for $q \geq q_0$, we can replace c by $c(c')^{q_0}$. In the domain case we can use this idea unless $I = (0)$. But then $I^* = (0)$, and we automatically have that $cu^q \in I^{[q]}$ for all q when $u \in I^*$, since $u = 0$.

We also note:

(3) *If $R \subseteq S$ are domains, and $I \subseteq R$ is an ideal, $I^* \subseteq (IS)^*$, where I^* is taken in R and $(IS)^*$ in S.*

This is immediate from the definition of tight closure, since nonzero elements of R map to nonzero elements of S and $I^{[q]} \subseteq (IS)^{[q]} = I^{[q]}S$. More generally, this holds when $R \to S$ is a homomorphism such that R° maps into S°. In fact, under mild conditions on the rings, for any map $R \to S$ (it need not be injective) the tight closure of every ideal $I \subseteq R$ maps into the tight closure of IS in S, but the proofs are difficult.

Note that Theorem on colon-capturing from p. 4 of the Lecture Notes of March 11 can now be re-stated as follows:

Theorem (colon-capturing). *Let A be an $\mathbb{N}$-graded domain finitely generated over a field K of prime characteristic $p > 0$. Let $F_1, \ldots, F_d$ be a homogeneous system of parameters for A. Then for $0 \leq i \leq d-1$, $(F_1, \ldots, F_i)A :_A F_{i+1} \subseteq (F_1, \ldots, F_i)^*$.* $\square$

We shall see that there is a local version of this result. Mild conditions on the local ring are needed: for the reader is familiar with the notion of "excellent" local ring, we note that being excellent suffices. It is also sufficient if the ring is a homomorphic image of a regular local ring or even of a Cohen-Macaulay local ring. Since we shall show that every complete local ring is a homomorphic image of a regular local ring, the result is valid in the complete case.

(4) *If A is a local domain of characteristic $p > 0$ that is a homomorphic image of a Cohen-Macaulay ring and $f_1, \ldots, f_d$ is a system of parameters for A, then for $1 \leq i \leq d-1$, $(f_1, \ldots, f_i)A :_A f_{i+1} \subseteq \big((f_1, \ldots, f_i)A\big)^*$.*

The proof is postponed.

We next note that the Lemma on p. 5 of the Lecture Notes of March 11 may now be stated as follows:

Lemma. *Every ideal of the polynomial ring* $K[x_1, \ldots, x_n]$ *over a field* K *of prime characteristic* $p > 0$ *is tightly closed.* $\square$

We shall eventually show the following:

(5) *If* R *is a regular Noetherian ring of characteristic* $p > 0$*, then every ideal of* R *is tightly closed.*

The key point in the proof is that the Frobenius endomorphism is flat for all regular rings of characteristic $p > 0$. We shall prove this making use of the structure theory of complete local rings.

We note that given a theory of tight closure satisfying conditions (1) — (5), one immediately gets the following:

Theorem. *Let* R *be a regular ring of characteristic* $p > 0$ *and let* $A \subseteq R$ *be a subring such that* A *is a direct summand of* R *as* A*-modules. Then* A *is Cohen-Macaulay.*

Sketch of proof, assuming (1) — (5). The issue is local on A. Assume that (A, m) is local. One may replace A by its completion and R by its completion at mR. Thus, we may assume that the Theorem on colon-capturing holds for A, i.e., that (4) holds. Let $f_1, \ldots, f_d$ be a system of parameters for A. Suppose $uf_{i+1} \in (f_1, \ldots, f_i)A$. Then $u \in \big((f_1, \ldots, f_i)A\big)^*$ by (4). By (3), we have that $u \in \big((f_1, \ldots, f_i)R\big)^*$. By (5), we have that $u \in (f_1, \ldots, f_i)R \cap A$. Since A is a direct summand of R, it follows that $u \in (f_1, \ldots, f_i)A$. Thus, $f_1, \ldots, f_d$ is a regular sequence in A, and A is Cohen-Macaulay. $\square$

Thus, the development of a sufficiently good tight closure theory in characteristic $p > 0$ yields a proof that direct summands of regular rings are Cohen-Macaulay.

There is also a theory of tight closure for Noetherian rings containing $\mathbb{Q}$ that has properties (1) — (5). It is defined in a convoluted way using reduction to positive characteristic p. In consequence, it is known that direct summands of regular rings are Cohen-Macaulay in equal characteristic 0. It remains an open question if the ring does not contain a field.

We shall also see that the existence of a good tight closure theory has many other applications.