Math 615, Winter 2020: Lecture of Wednesday, April 1

Another use of tight closure:
contracted expansions from module-finite extension rings

In this lecture we discuss the problem of understanding $IS \cap R$ when R is a Noetherian domain and S is a module-finite or integral extension. If R contains the rational numbers $\mathbb{Q}$ and is normal, the situation is simple: it turns out that $IS \cap R = I$. In characteristic $p > 0$ the problem is much more difficult. Even when R is normal, $IS \cap R$ can be strictly larger than I. But we shall show that $IS \cap R \subseteq I^*$, which provides another use for tight closure. One can also consider the union of the ideals $IS \cap R$ as S runs through all module-finite extensions of R (it turns out not to matter whether the extension is also a domain), which gives a new kind of closure of I, called the *plus* closure I^+ such that $I \subseteq I^+ \subseteq I^*$ in positive characteristic p.

We also discuss the status of several questions about tight closure that were either open questions for a considerable length of time and then resolved, or that continue to be open questions.

Let R be a domain. Suppose that $R \subseteq S$ is a module-finite extension. In general, $I \subseteq IS \subseteq R$, but $IS \cap R$ may be larger than I. The main case is where S is also a domain. For S has a minimal prime $\mathfrak{p}$ disjoint from the multiplicative system $R - \{0\}$, and R injects into $\overline{S} = S/\mathfrak{p}$, which is a domain module-finite over R. Moreover, if $r \in R$ is in IS, then the image of r in $S/\mathfrak{p}$ is in $I\overline{S}$.

Suppose that $f \in R$, $g \in R - \{0\}$, and f/g is integral over R but not in R, which means that $f \notin gR$. We may take $S = R[f/g]$. Then $f \in gS \cap R - gR$, so that when R is not normal even principal ideals fail to be contracted from module-finite extensions. But if R is normal and contains $\mathbb{Q}$, then every ideal is contracted from every module-finite extension S. To see this, first note that it suffices to consider the case where S is a domain, by the argument above. Let $\mathcal{K}$ and $\mathcal{L}$ be the respective fraction fields of R and S. Multiplication by an element of $\mathcal{L}$ gives a map $\mathcal{L} \to \mathcal{L}$ which is $\mathcal{K}$-linear. If we simply think of this map as an endomorphism of the finite-dimensional $\mathcal{K}$-vector space $\mathcal{L}$, we may take its trace: i.e., pick a basis for $\mathcal{L}$ over $\mathcal{K}$, and take the sum of the diagonal entries of the matrix of the multiplication map with respect to this basis. This is independent of the choice of basis.

This trace map $\mathrm{Tr}_{\mathcal{L}/\mathcal{K}} : \mathcal{L} \to \mathcal{K}$ is $\mathcal{K}$-linear (hence, R-linear) and has value h on 1, where $h = [\mathcal{L} : \mathcal{K}]$. When R is a normal Noetherian ring, it turns out that the values of this map on S are in R. (One can see this as follows. First, R is the intersection of its localizations R_P at height one primes P. For if $f, g \in R$, $g \neq 0$, and f/g is in the fraction field of R but not in R, then $f \notin gR$. The associated primes of gR have height one, because R is normal. Using the primary decomposition of gR, we see that $f \notin \mathfrak{A}$ for some ideal $\mathfrak{A}$ primary to an associated P of gR of height one, and since elements of $R - P$ are not zerodivisors on $\mathfrak{A}$, $f \notin \mathfrak{A}R_P$ and so $f \notin gR_P$, i.e., $f/g \notin R_P$. If $Tr_{\mathcal{L}/\mathcal{K}}$ has a value on S not in R, we may preserve this while localizing at a height one prime P of R. But then we may replace

R, S by R_P, S_P and assume that $R = R_P$ is a Noetherian discrete valuation ring. Since S is a torsion-free module over R, it is free, and has a free basis over R, say $s_1, \ldots, s_j$, consisting of elements of S. This is also a basis for $\mathcal{L}$ over $\mathcal{K}$, and can be used to calculate the trace of s. But now the matrix for multiplication by s has entries in R: for every s_i we have

$$ss_i = \sum_{j=1}^{h} r_{ij}s_j$$

with the $r_{ij} \in R$. But then the trace is $\sum_{i=1}^{h} r_{ii}$ and is in R after all. The condition that R be Noetherian is not really needed: for example, in the general case, an integrally closed domain can be shown to be a directed union of Noetherian integrally closed domains, from which the general case can be deduced. There are several other lines of argument.)

Finally, $\frac{1}{h}\mathrm{Tr}_{\mathcal{L}/\mathcal{K}} : S \to R$ splits $R \hookrightarrow S$ as a map of R-modules: by R-linearity, the fact that 1 maps to itself implies that the same holds for every element of R. Since we have a splitting, it follows that every ideal of R is contracted from S.

Although ideals are contracted from module finite-extensions of normal Noetherian domains that contain $\mathbb{Q}$, this is false in positive characteristic p.

Example. Let $R = K[X, Y, Z]/(X^3+Y^3+Z^3)$ where K is a field of characteristic 2. Then $z^2 \notin (x, y)R$, as noted earlier. But if we make a module-finite domain extension S of R that contains $x^{1/2}$, $y^{1/2}$, and $z^{1/2}$, then since $z^3 = x^3 + y^3$ (we are in characteristic 2, so that minus signs are not needed) we have $z^{3/2} = x^{3/2} + y^{3/2}$ (since squaring commtutes with addition and elements have at most one square root in domains of characteristic 2, taking square roots also commutes with addition in domains of characteristic 2). But then

$$z^2 = z^{1/2}z^{3/2} = z^{1/2}(xx^{1/2} + yy^{1/2}) = x^{1/2}z^{1/2}x + y^{1/2}z^{1/2}y \in (x, y)S \cap R - R.$$

However, tight closure "captures" the contracted expansion to a module-finite extension, which gives another proof that $z^2 \in (x, y)^*$ in the Example just above.

Theorem. *Let R be a Noetherian domain, and let S be any integral extension of R. Then for every ideal I of R, $IS \cap R \subseteq I^*$.*

Proof. Suppose that $f \in R$ and

$$(*) \quad f = \sum_{i=1}^{h} f_j s_j$$

where the $f_j \in I$ and the $s_j \in S$. We may replace S by $R[s_1, \ldots, s_h] \subseteq S$, and so assume that S is module-finite over R. Second, we may kill a minimal prime of S disjoint from $R - \{0\}$ and so assume that S is a module-finite domain extension of R. Choose a maximal set of R-linearly independent elements of S, say $u_1, \ldots, u_k$, so that $Ru_1 + \cdots + Ru_k$ is R-torsion. It follows that some nonzero element $r \in R$, we have that

$$S \cong rS \subseteq Ru_1 + \cdots + Ru_k.$$

Thus, we have an embedding $S \hookrightarrow R^k$. Suppose that $1 \in S$ has as its image in R^k an element whose i th coordinate is nonzero, so that the composite map $S \hookrightarrow R^k \xrightarrow{\pi_i} R$ is nonzero on the element $1 \in S$, where π_i is the i th coordinate projection of $R^k \twoheadrightarrow R$. This gives an R-linear map $\theta : S \to R$ such that $\theta(1) = c \in R$ is nonzero. Now take q th powers of both sides of $(*)$, yielding

$$(**) \quad f^q \cdot 1 = \sum_{i=1}^{h} f_j^q s_j^q.$$

Since θ is R-linear and $f, f_1, \ldots, f_h \in R$, this yields

$$f^q\theta(1) = \sum_{i=1}^{h} f_j^q \theta(s_j^q),$$

and so $cf^q \in I^{[q]}$ for all q. This implies that $f \in I^*$. $\square$

Open questions: tight closure, plus closure, and localization

We want to consider some open questions in tight closure theory, and some related problems about when rings split from their module-finite extension algebras. After we do this, we shall prove some specific results in the characteristic p theory. It will turn out that to proceed further, we will need the structure theory of complete local rings, which we will develop next.

One of the longest standing and most important questions about tight closure is when tight closure commutes with localization. E.g., if R is Noetherian of prime characteristic $p > 0$, I is an ideal of R, and W is a multiplicative system of R, when is $W^{-1}(I_R^*)$ the same as $(W^{-1})^*_{W^{-1}R}$? It is easy to prove that $W^{-1}(I_R^*) \subseteq (W^{-1})^*_{W^{-1}R}$. This was an open question for more than twenty years. It is known to be true in many cases, but false in general, by a result of [H. Brenner and P. Monsky, See, for example, [I. Aberbach, M. Hochster, and C. Huneke, *Localization of tight closure and and modules of finite phantom projective dimension*, J. Reine Angew. Math. (Crelle's Journal) **434** (1993), 67–114], and [M. Hochster and C. Huneke, *Test exponents and localization of tight closure*, Michigan Math. J. **48** (2000), 305–329] for a discussion of the problem.

We saw in the Theorem proved on p. 168 of this lecture that tight closure "captures" contracted extension from module-finite and even integral extensions. We shall add this as (6) to our list of desirable properties for a tight closure theory, which becomes the following:

(0) *$u \in N_M^*$ if and only if the image $\overline{u}$ of u in M/N is in $0^*_{M/N}$.*

(1) *N_M^* is a submodule of M and $N \subseteq N_M^*$. If $N \subseteq Q \subseteq M$ then $N_M^* \subseteq Q_M^*$.*

(2) If $N \subseteq M$, then $(N_M^*)_M^* = N_M^*$.

(3) *If $R \subseteq S$ are domains, and $I \subseteq R$ is an ideal, $I^* \subseteq (IS)^*$, where I^* is taken in R and $(IS)^*$ in S.*

(4) *If A is a local domain then, under mild conditions on A (the class of rings allowed should include local rings of a finitely generated algebra over a complete local ring or over $\mathbb{Z}$), and $f_1, \ldots, f_d$ is a system of parameters for A, then for $1 \leq i \leq d-1$, $(f_1, \ldots, f_i)A :_A f_{i+1} \subseteq \bigl((f_1, \ldots, f_i)A\bigr)^*$.*

(5) If R is regular, then $I^* = I$ for every ideal I of R.

(6) For every module-finite extension ring R of S and every ideal I of R, $IS \cap R \subseteq I^*$.

These are all properties of tight closure in prime characteristic $p > 0$, and also of the theory of tight closure for Noetherian rings containing $\mathbb{Q}$ that we described in the Lecture of March 30. In characteristic $p > 0$, (4) holds for homomorphic images of Cohen-Macaulay rings, and for excellent local rings. If $R \supseteq \mathbb{Q}$, (4) holds if R is excellent. We will prove that (4) holds in prime characteristic for homomorphic images of Cohen-Macaulay rings quite soon. We have proved (5) in prime characteristic $p > 0$ for polynomial rings over a field, but not yet for all regular rings. To give the proof for all regular rings we need to prove that the Frobenius endomorphism is flat for all such rings, and we shall eventually use the structure theory of complete local rings to do this.

An extremely important open question is whether there exists a closure theory satisfying (1) — (6) for Noetherian rings that need not contain a field.

The Theorem proved on p. 168 of this lecture makes it natural to consider the following variant notion of closure. Let R be any integral domain. Let R^+ denote the integral closure of R in an algebraic closure $\overline{\mathcal{K}}$ of its fraction field $\mathcal{K}$. We refer to this ring as the *absolute integral closure* of R. R^+ is unique up to non-unique isomorphism, just as the algebraic closure of a field is. Any module-finite (or integral) extension domain S of R has fraction field algebraic over $\mathcal{K}$, and so S embeds in $\overline{\mathcal{K}}$. It follows that S embeds in R^+, since the elements of S are integral over R. Thus, R^+ contains an R-subalgebra isomorphic to any other integral extension domain of R: it is a maximal extension domain with respect to the property of being integral over R. R^+ is the directed union of its finitely generated subrings, which are module-finite over R. R^+ is also charactized as follows: it is a domain that is an integral extension of R, and every monic polynomial with coefficients in R^+ factors into monic linear polynomials over R^+.

Given an ideal $I \subseteq R$, the following two conditions on $f \in R$ are equivalent:

(1) $f \in IR^+ \cap R$.

(2) For some module-finite extension S of R, $f \in IS \cap R$.

The set of such elements, which is $IR^+ \cap R$, is denoted I^+, and is called the *plus closure* of I. (The definition can be extended to modules $N \subseteq M$ by defining N^+_M to be the kernel of the map $M \to R^+ \otimes_R (M/N)$.)

By the Theorem on p. 168 of the notes for this lecture, which is property (6) above in characteristic $p > 0$, we have that

$$I \subseteq I^+ \subseteq I^*$$

in prime characteristic $p > 0$. Whether $I^+ = I^*$ in general under mild conditions for Noetherian rings of prime characteristic $p > 0$ is another very important open question. It is not known to be true even in finitely generated algebras of Krull dimension 2 over a field.

However, there are some substantial positive results. It is known that under the mild conditions on the local domain R (e.g., when R is excellent), if I is generated by part of a system of parameters for R, then $I^+ = I^*$. See [K. E. Smith, *Tight closure of parameter ideals*, Inventiones Math. **115** (1994) 41–60]. Moreover, H. Brenner [H. Brenner, *Tight closure and plus closure in dimension two*, Amer. J. Math. **128** (2006) 531–539] proved that if R is the homogeneous coordinate ring of a smooth projective curve over the algebraic closure of $\mathbb{Z}/p\mathbb{Z}$ for some prime integer $p > 0$, then $I^* = I^+$ for homogeneous ideals primary to the homogeneous maximal ideal. In [G. Dietz, *Closure operations in positive characteristic and big Cohen-Macaulay algebras*, Thesis, Univ. of Michigan, 2005] the condition that the ideal be homogeneous is removed: in fact, there is a corresponding result for modules $N \subseteq M$ when M/N has finite length. Brenner's methods involve the theory of semi-stable vector bundles over a smooth curve (in fact, one needs the notion of a *strongly* semi-stable vector bundle, where "strongly" means that the bundle remains semi-stable after pullback by the Frobenius map).

One reason for the great interest in whether plus closure commutes with tight closure is that it is known that plus closure commutes with localization. Hence, if $I^* = I^+$ in general (under mild conditions on the ring) one gets the result that tight closure commutes with localization.

The notion of plus closure is of almost no help in understanding tight closure when the ring contains the rationals. The reason for this is the result established using field trace in the first two pages of the notes for this lecture, which we restate formally here.

Theorem. *Let R be a normal Noetherian domain with fraction field $\mathcal{K}$ and let S be a module-finite extension domain with fraction field $\mathcal{L}$. Let $h = [\mathcal{L} : \mathcal{K}]$. If $\mathbb{Q} \subseteq R$, or, more generally, if h has an inverse in R, then $\frac{1}{h}\mathrm{Tr}_{\mathcal{L}/\mathcal{K}}$ gives an R-module retraction $S \to R$.* □

It follows that if $\mathbb{Q} \subseteq R$ and R is a normal domain, then $I^+ = I$ for every ideal I of R. Many normal rings (in some sense most normal rings) that are essentially of finite type over $\mathbb{Q}$ are not Cohen-Macaulay, and so contain parameter ideals that are not tightly closed. This shows that plus closure is not a greatly useful notion in Noetherian domains that contain $\mathbb{Q}$.