Math 615, Winter 2020: Lecture of Friday, April 3

In this lecture we begin discussion of *weakly F-regular rings*: these are the Noetherian rings of characteristic $p > 0$ such that every ideal is tightly closed. In particular, regular rings are weakly F-regular. It will turn out that R is weakly F-regular if and only if its localization at every *maximal ideal* is weakly F-regular. There are many examples when the ring is not regular. See the Examples on the next page. It is not known even for very good rings (finitely generated algebras over an algebraically closed field, for example) whether weak F-regularity is preserved by localization at an arbitrary prime. If that is true, it is preserved by localization at any multiplicative system. It is conjectured that under mild conditions on R, the two notions should be equivalent: the question has been open for over thirty-three years. Weakly F-regular rings are automatically normal and, under very mild assumptions, Cohen-Macaulay. We have mostly talked about normal domains in the past. More generally, a finite product of normal domains is also called *normal*. With this definition, the problem of whether a ring is normal is local: R is normal if and only if all local rings of R are normal domains.

We also define a *splinter* to be a Noetherian domain that splits, as a module over itself, from every module-finite ring extension. This condition turns out to imply normality, and, for rings containing $\mathbb{Q}$, it characterizes the splinters. In characteristic p, this condition holds if the ring is weakly F-regular, and it is known to be equivalent to weak F-regularity for Cohen-Macaulay rings whose local rings have type 1 (these are called *Gorenstein rings*).

Weakly F-regular rings and F-regular rings

We define a Noetherian ring R of prime characteristic $p > 0$ to be *weakly F-regular* if every ideal is equal to its tight closure, i.e., every ideal is tightly closed. We define R to be *F-regular* if all of its localizations are weakly F-regular. It is not known whether weakly F-regular implies F-regular, even for domains finitely generated over a field. This would follow if tight closure were known to commute with localization.

We have already proved that polynomial rings over a field of positive characteristic are weakly F-regular, and we shall prove that every regular ring of positive characteristic is F-regular. This is one reason for the terminology. The "F" suggests the involvement of the Frobenius endomorphism.

We shall soon show that a weakly F-regular ring is normal, and, if it is a homomorphic image of a Cohen-Macaulay ring, is itself Cohen-Macaulay.

Theorem. *A direct summand A of a weakly F-regular domain is weakly F-regular, and a direct summand of an F-regular domain is F-regular.*

Proof. Assume that R is weakly F-regular. If $f \in I_A^*$, then $f \in (IR)^* \cap A = IR \cap A = I$. Since the direct summand condition is preserved by localization on A, it follows that a direct summand of an F-regular domain is F-regular. $\square$

Examples of F-regular rings. Fix a field K of characteristic $p > 0$. Normal rings finitely generated over K by monomials are direct summand of regular rings, and so are F-regular. If X is an $r \times s$ matrix of indeterminates over K with $1 \leq t \leq r \leq s$, then it is known that $K[X]/I_t(X)$ is F-regular, and that the ring generated by the $r \times r$ minors of X over K is F-regular (this is the homogeneous coordinate ring of the Grassmann variety). See [M. Hochster and C. Huneke, *Tight closure of parameter ideals and splitting in module-finite extensions*, J. of Algebraic Geometry **3** (1994) 599–670], Theorem (7.14). We have already observed that these rings are direct summands of polynomial rings when K has characteristic 0, but this is not true in any obvious way when the characteristic is positive.

Splitting from module-finite extension rings

It is natural to attempt to characterize the Noetherian domains R such that R is a direct summand, as an R-module, of every module-finite extension ring S. We define a Noetherian domain R with this property to be a *splinter*. We then have the following result, which was actually proved in the preceding lecture, although it was not made explicit there.

Theorem. *Let R be a Noetherian domain.*

(a) *If R is a splinter, then every ideal of R is contracted from every integral extension.*

(b) *If R is a splinter, then R is normal.*

(c) *R is a splinter if and only if it is a direct summand of every module-finite domain extension.*

(d) *If $\mathbb{Q} \subseteq R$, then R is a splinter if and only if R is normal.*

Proof. For part (a), suppose $f,\, f_1,\, \ldots\, , f_h \in R$ and $f = \sum_{i=1}^{h} f_i s_i$ with the s_i in S. Then we have the same situation when S is replaced by $R[s_1,\, \ldots\, , s_h]$. Hence, it suffices to show that every ideal of R is contracted from every module-finite extension S. But then we have an R-linear retraction $\phi : S \to R$, and the result is part (a) of the Lemma at the top of p. 2 of the Lecture of March 30.

Part (b) has already been established in the fourth paragraph on p. 159 of the Lecture of March 30.

For part (c), we have already observed that S has a minimal prime $\mathfrak{p}$ disjoint from $R - \{0\}$, and it suffices to split the injection $R \hookrightarrow S/\mathfrak{p}$.

Finally, for part (d), the existence of the required splitting when S is a domain is proved at the bottom of p. 4 and top of p. 5 of the Lecture Notes of March 30, using field trace, and restated on p. 3 here. $\square$

The example on p. 5 of the Lecture Notes of March 30 shows that in positive characteristic p, a normal domain need not be a splinter. The property of being a splinter in characteristic p is closely related to the property of being weakly F-regular.

We first note the following fact: we shall not give the proof in these lectures, but refer the reader to [M. Hochster, *Contracted ideals from integral extensions of regular rings*, Nagoya Math. J. **51** (1973) 25–43] and [M. Hochster, *Cyclic purity versus purity in excellent Noetherian rings*, Trans. Amer. Math. Soc. **231** (1977) 463–488].

Theorem. *Let R be a normal Noetherian domain. Then R is a direct summand of a module-finite extension of S if and only if every ideal of R is contracted from S.*

Of course, we know the "only if" part.

Corollary. *Let R be a normal Noetherian domain of positive characteristic p. Then R is a splinter if and only if for every ideal $I \subseteq R$, $I = I^+$.*

Corollary. *If R is a normal Noetherian domain and R is weakly F-regular, then R is a splinter.*

Proof. This is immediate from the preceding result, since $I^+ \subseteq I^*$. □

We shall see quite soon that if R is weakly F-regular it is automatic that R is normal. If plus closure is the same as tight closure, then it would follow that R is weakly F-regular if and only if R is a splinter. This is an open question.

We have already observed that in characteristic $p > 0$, regular rings are weakly F-regular, although we have not prove this. Assuming this for the moment we have:

Corollary. *A regular ring that contains a field is a direct summand of every module-finite extension ring.*

This was conjectured by the author in 1969, and was been open question for regular rings that do not contain a field, such as polynomial rings over the integers, for 50 years. The case of dimension 3 was settled affirmatively in [R. C. Heitmann, *The direct summand conjecture in dimension three*, Annals of Math. (2) **156** (2002) 695–712]. The general case was settled by Y. André in 2016, and then a simpler proof was given by B. Bhatt.

It is also a major open question whether there exists a tight closure theory satisfying conditions (0) — (6) of p. 1 for Noetherian rings that need not contain a field. The existence of such a theory would imply that direct summands of regular rings are Cohen-Macaulay in general, and that regular rings are direct summands of all of their module-finite extensions in general. Such a theory would also settle many other open questions.