Math 615, Winter 2020: Lecture of Monday, April 6

In the first part of this lecture we consider only Noetherian rings of positive prime characteristic p. We prove that the tight closure of (0) is the nilradical, and we can conclude that weakly F-regular rings are reduced. We also study tight closure in products of rings, and prove that a finite product of weakly F-regular rings is weakly F-regular. We prove that if every principal ideal of a ring is tightly closed, then the ring is a finite product of normal domains (we adjust terminology, and call these finite products of normal domains *normal* as well.

The second part of the lecture reviews and refines properties of Cohen-Macaulay rings, especially those connected with chain conditions, and introduces the notion of *universally catenary*: R is universally catenary if every finitely generated R-algebra (this includes homomorphic images) is catenary. Cohen-Macaulay rings are universally catenary.

More on tight closure, weak F-regularity, and the Cohen-Macaulay property

We next want to study weakly F-rings, i.e., Noetherian rings of prime characteristic $p > 0$ such that every ideal is tightly closed. Until further notice, all given rings R are assumed to be Noetherian, of prime characteristic $p > 0$.

Proposition. *The tight closure of the* (0) *ideal in* R *is the ideal of all nilpotent elements. Hence, if* $(0) = (0)^*$*, the* R *is reduced. In particular, every weakly F-regular ring is reduced.*

Proof. If u is nilpotent then $1 \cdot u^q = 0$ for all $q \gg 0$. Conversely, if $c \in R^\circ$ and $cu^q = 0$ for all $q \gg 0$, then for every minimal prime $\mathfrak{p}$ we have that $cu^q \in \mathfrak{p}$ for some q. Since $c \notin \mathfrak{p}$, we have that $u^q \in \mathfrak{p}$ and so $u \in \mathfrak{p}$. But the intersection of the minimal primes is the set of nilpotent elements of R, and so u is nilpotent. The remaining statements are now obvious. $\square$

Proposition. *Suppose that* $R = S \times T$ *is a product ring, with* $S, T \neq 0$*. Then for every ideal* $I \times J$ *of* $S \times T$*, where* $I \subseteq S$ *and* $J \subseteq T$ *are ideals,* $(I \times J)^*_R = I^*_S \times J^*_T$.

Proof. The first point is that $(S \times T)^\circ = (S^\circ) \times (T^\circ)$. Hence if $cs^q \in I^{[q]}$ for all $q \gg 0$ and $dt^q \in J^{[q]}$ for all $q \gg 0$, we have that

$$(c,\, d)(s,\, t)^q \in I^{[q]} \times J^{[q]} = (I \times J)^{[q]}$$

for all $q \gg 0$. The converse is also immediate. $\square$

Corollary. *A finite product* $R_1 \times \cdots \times R_h$ *is weakly F-regular if and only if every factor is weakly F-regular.* $\square$

Theorem. *If every principal ideal of R is tightly closed, then R is a product of normal domains.*

Proof. The fact that $(0) = (0)^*$ implies that R is reduced. We first show that R is a product of domains. If there are two or more minimal primes, the minimal primes can be partitioned into two nonempty sets. Call the intersection of one set I and the intersection of the other set J. Then $I \cap J = 0$, and $I + J$ is not contained in any minimal prime $\mathfrak{p}$, for otherwise, $\mathfrak{p}$ would have to contain both a minimal prime of I and a minimal prime of J, and would be equal to both of these. Hence we can choose $f \in I$ and $g \in J$ such that $f + g$ is not in any minimal prime of R, and so is a nonzerodivisor. Note that $fg \in I \cap J$, and so $fg = 0$. Now

$$(f + g)f^q = f^{q+1} = f(f + g)^q$$

for all q, so that $f \in (f + g)^* = (f + g)R$. Thus, we can choose $r \in R$ such that $f = r(f + g) = rf + rg$, and the $f - rf = rg$. Since $f \in I$ and $g \in J$, both sides must vanish, and so $f = rf$ and $rg = 0$. Now $r(f + g) = rf = f$, and

$$r^2(f + g) = r(rf + rg) = r(f + 0) = rf = f,$$

so that

$$(f + g)(r^2 - r) = 0.$$

Since $f + g$ is not a zerodivisor, we have that $r^2 - r = 0$. Since $rf = f$ is not 0 (or $f + g$ would be in the minimal primes containing g) $r \neq 0$. Since $rg = 0$, $r \neq 1$. Therefore, R contains a non-trivial idempotent, and is a product of two rings. Both have the property that principal ideals are tightly closed, because a principal ideal of $S \times T$ is the product of a principal ideal of S and a principal ideal of T, and we may apply the Proposition above.

We may apply this argument repeatedly and so write R as a finite product of rings with the property that every principal ideal is tightly closed, and such that none of the factors is a product. Each of the factors must have just one minimal prime, and so is a domain. It remains to see that if principal ideals are tightly closed in a domain R, then R is normal. Suppose that $f, g \in R$, $g \neq 0$, and f/g is integral over R. Let $S = R[f/g]$, which is module-finite over R. Then $f = g(f/g) \in gS$, and so $f \in (gR)^*$. But $(gR)^* = gR$, and so $f \in gR$, i.e., $f/g \in R$, as required. $\square$

We next want to show that, under mild conditions on R, if R is weakly F-regular then R is Cohen-Macaulay. To prove this, we will need to generalize the results on colon-capturing that we have already obtained in finitely generated $\mathbb{N}$-graded algebras over a field.

We first review some facts about Cohen-Macaulay rings. This material is in the Lecture of March 11.

We recall that if (R, m, K) is a Cohen-Macaulay local ring, then for every minimal prime $\mathfrak{p}$ of R, $\dim(R/\mathfrak{p}) = \dim(R)$.

Thus, a Cohen-Macaulay local ring cannot exhibit the kind of behavior one observes in $R = K[[x, y, z]]/\big((x, y) \cap (z)\big)$: this ring has two minimal primes. One of them, $\mathfrak{p}_1$, generated by the images of x and y, is such that $R/\mathfrak{p}_1$ has dimension 1. The other, $\mathfrak{p}_2$, generated by the image of z, is such that $R/\mathfrak{p}_2$ has dimension 2.

We also recall that a Noetherian ring is called *catenary* if for any two prime ideals $P \subseteq Q$, any two saturated chains of primes joining P to Q have the same length. If R is catenary, then so is R/I for every ideal I, since primes containing I are in bijective correspondence with primes of R containing I, and saturated chains of primes in R/I joining P/I to Q/I, where $I \subseteq P \subseteq Q$ and P, Q are primes of R, correspond to saturated chains of primes of R joining P to Q. Similarly, any localization of a catenary ring is catenary. M. Nagata gave the first examples of Noetherian rings that are not catenary: there is a local domain (R, m, K) of dimension 3, for example, containing saturated chains $0 \subset Q \subset m$ and $0 \subset P_1 \subset P_2 \subset m$, where all inclusions are strict. See [M. Nagata, *Local rings*, Interscience, New York, 1962], Appendx A1, pp. 204–205. Although Q has height one and $\dim(R) = 3$, the dimension of R/Q is 1. Nagata also showed that even when a Noetherian ring is catenary, the polynomial ring in one variable over it need not be.

A Noetherian ring R is called *universally catenary* if every finitely generated R-algebra is catenary. This is equivalent to assuming that all polynomial rings in finitely many variables over R are catenary, since all finitely generated R-algebras are homomorphic images of such polynomial rings, and a homomorphic image of a catenary ring is catenary (because, if $I \subseteq P \subseteq Q$, the chains of primes between P and Q correspond bijectively with the chains of primes between P/I and Q/I in R/I). Note also that , similarly, all localizations of catenary rings are catenary, and it follows easily that localizations of universally catenary rings are universally catenary. Cohen-Macaulay rings are universally catenary, as we show in the two results below. The following result strengthens a bit what we know about about chains of primes in Cohen-Macaulay rings.

Theorem. *A Cohen-Macaulay ring R is catenary, and for any two prime ideals $P \subseteq Q$ in R, every saturated chain of prime ideals joining P to Q has length* $\mathrm{height}\,(Q) - \mathrm{height}\,(P)$.

Proof. For the first part, the issues are unaffected by localizing at Q. Thus, we may assume that R is local and that Q is the maximal ideal. There is part of a system of parameters of length $h = \mathrm{height}\,(P)$ contained in P, call it $x_1, \ldots, x_h$. This sequence is a regular sequence on R and in so on R_P, which implies that its image in R_P is system of parameters. We now replace R by $R/(x_1, \ldots, x_h)$. Both the dimension and depth of R have decreased by h, so that R is still Cohen-Macaulay. Q and P are replaced by their images, which have heights $\dim(R) - h$ and 0, and $\dim(R) - h = \dim(R/(x_1, \ldots, x_h)$. We have therefore reduced to the case where R is local and P is a minimal prime. We know that $\dim(R) = \dim(R/P)$, and so at least one saturated chain from P to Q has length $\mathrm{height}\,(Q) - \mathrm{height}\,(P) = \mathrm{height}\,(Q) - 0 = \dim(R)$. To complete the proof, it will suffice to show that all saturated chains from P to Q have the same length, and we may use induction on $\dim(R)$. Consider two such chains, and let their smallest elements other than P be P_1 and P_1'. Choose an element x in P_1 not in any minimal prime, and an element y of P_1' not in any minimal prime. Then xy is a nonzerodivisor in R, and P_1, P_1' are both minimal primes of xy. The ring $R/(xy)$ is Cohen-Macaulay of dimension $\dim(R) - 1$. The result now follows from the induction hypothesis applied to $R/(xy)$: the images of the two saturated chains (omitting P from each) give saturated chains joining $P_1/(xy)$ (respectively, $P'_1/(xy)$) to $Q/(xy)$ in $R/(xy)$. These have the same length, and, hence, so did the original two chains. □

Corollary. *Cohen-Macaulay rings are universally catenary, i.e., a finitely generated algebra over a Cohen-Macaulay ring is catenary.*

Proof. Such an algebra is a homomorphic image of a polynomial ring in finitely many variables over a Cohen-Macaulay ring, which is again Cohen-Macaulay, and homomorphic images of catenary rings are catenary. □