Math 615, Winter 2020: Lecture of Wednesday, April 8

Throughout this lecture we will work with Noetherian rings of positive prime characteristic p. The first main goal is to prove a result on colon-capturing for tight closure in rings that are homomorphic images of Cohen-Macaulay rings. The condition that a ring be a homomorphic image of a Cohen-Macaulay ring is not very restrictive: it typically holds for the Noetherian rings that come up in algebraic geometry, algebraic combinatorics, several complex variables (e.g., convergent power series rings) and algebraic number theory. In fact, the rings that come up are usually homomorphic images of regular rings. The property of being a homomorphic image of a Cohen-Macaulay ring passes to finitely generated algebras and to all localizations at multiplicative systems. Note that every ring that is finitely generated either over a field or over a complete local ring has this property (as we shall see later, every complete local ring is a homomorphic image of a complete regular local ring).

The argument for colon-capturing needs a preliminary lemma on prime avoidance for cosets, which is used in the proof of another lemma on lifting systems of parameters from quotients S/P to S.

It is then shown that a ring is weakly F-regular if and only if all of its localizations at *maximal* ideals are weakly F-regular. We emphasize that it is *not* known whether the localizations of a weakly F-regular ring at non-maximal primes must again be weakly F-regular (the stronger property defines F-*regularity* without the modifier "weakly"). Coupled with the colon-capturing result, this proves that every weakly F-regular ring that is a homomorphic image of a Cohen-Macaulay ring is Cohen-Macaulay.

Here are the two preliminary results needed for the theorem on colon-capturing.

Lemma (prime avoidance for cosets). *Let S be any commutative ring, $x \in S$, $I \subseteq S$ an ideal and $P_1, \ldots, P_k$ prime ideals of S. Suppose that the coset $x + I$ is contained in $\bigcup_{i=1}^k P_i$. Then there exists j such that $Sx + I \subseteq P_j$.*

Proof. If $k = 1$ the result is clear. Choose $k \geq 2$ minimum giving a counterexample. Then no two P_i are comparable, and $x + I$ is not contained in the union of any $k - 1$ of the P_i. Now $x = x + 0 \in x + I$, and so x is in at least one of the P_j: say $x \in P_k$. If $I \subseteq P_k$, then $Sx + I \subseteq P_k$ and we are done. If not, choose $i_0 \in I - P_k$. We can also choose $i \in I$ such that $x + i \notin \bigcup_{j=1}^{k-1} P_i$. Choose $u_j \in P_j - P_k$ for $j < k$, and let u be the product of the u_j. Then $ui_0 \in I - P_k$, but is in P_j for $j < k$. It follows that $x + (i + ui_0) \in x + I$, but is not in any P_j, $1 \leq j \leq k$, a contradiction. $\square$

Lemma. *Let S be a Cohen-Macaulay local ring, let P be a prime ideal of S of height h, and let $x_1, \ldots, x_{i+1}$ be part of a system of parameters of $R = S/P$. Let $y_1, \ldots, y_h \in P$ be part of a system of parameters for S (we have a regular sequence on S of length h since* *the depth of S on P is the height of P, and this will be part of a system of parameters). Then there exist elements*

$\widetilde{x}_1, \dots, \widetilde{x}_{i+1}$ *of S such that*

$\widetilde{x}_j$ *maps to x_j modulo P, $1 \le j \le i+1$, and $y_1, \dots, y_h, \widetilde{x}_1, \dots, \widetilde{x}_{i+1}$ is part of a system of parameters for S.*

Proof. We construct the $\widetilde{x}_j$ recursively. Suppose that the $\widetilde{x}_j$ for $j < k+1 \le i+1$ have been chosen so that $y_1, \dots, y_h, \widetilde{x}_1, \dots, \widetilde{x}_k$ is part of a system of parameters for S. Here, k is allowed to be 0 (i.e., we may be choosing $\widetilde{x}_1$). We want to choose an element of $x_{k+1} + P$ that is not in any minimal prime of $y_1, \dots, y_h, \widetilde{x}_1, \dots, \widetilde{x}_k$, and these all have height at most $h+k$. By the Lemma on prime avoidance for cosets, if $\widetilde{x}_{k+1} + P$ is contained in the union, then $Sx_{k+1} + P$ is contained in one of them, say Q. Working modulo P we have that Q/P is a minimal prime $x_1, \dots, x_{k+1}$ of height at most $h+k-h = k$. This is a contradiction, since $x_1, \dots, x_{k+1}$ is part of a system of parameters in S/P, and so any minimal prime must have height at least $k+1$. $\square$

Theorem (colon-capturing). *Let (R, m, K) be a local domain of prime characteristic $p > 0$, and suppose that R is a homomorphic image of a Cohen-Macaulay ring of characteristic p. Let $x_1, \dots, x_{i+1}$ by part of a system of parameters in R. Then*

$$(x_1, \dots, x_i) :_R x_{i+1} \subseteq (x_1, \dots, x_i)^*.$$

Proof. Suppose that $R = S/P$, where S is Cohen-Macaulay of characteristic p, and let Q be the inverse image of m in S. Then R is also a homomorphic image of S_Q, since $S_Q/PS_Q \cong (S/P)_Q = R_Q = R_m = R$. Hence, we may assume that S is local. Choose $y_1, \dots, y_h$ and $\widetilde{x}_1, \dots, \widetilde{x}_{i+1}$ as in the preceding Lemma. Since P is a minimal prime of $(y_1, \dots, y_h)$ in S, we can choose $\widetilde{c} \in S - P$ and an integer $N > 0$ such that $\widetilde{c}P^N \in (y_1, \dots, y_h)S$. Let $c \neq 0$ be the image of $\widetilde{c}$ in R. Suppose that $fx_{i+1} = f_1x_1 + \cdots + f_ix_i$ in R. Then we can choose elements $\widetilde{f}$ and $\widetilde{f}_1, \dots, \widetilde{f}_i$ in S that lift f and $f_1, \dots, f_i$ respectively to S. This yields an equation

$$\widetilde{f}\widetilde{x}_{i+1} = \widetilde{f}_1\widetilde{x}_1 + \cdots + \widetilde{f}_i\widetilde{x}_i + \Delta$$

in S, where $\Delta \in P$. Then for all $p^e = q \ge N$ we have

$$\widetilde{f}^q\widetilde{x}_{i+1}^q = \widetilde{f}_1{}^q\widetilde{x}_1^q + \cdots + \widetilde{f}_i^q\widetilde{x}_i^q + \Delta^q$$

We may multiply both sides by $\widetilde{c}$, and use the fact that $\widetilde{c}\Delta^q \in cP^N \subseteq (y_1, \dots, y_h)$ to conclude that

$$(*) \quad \widetilde{c}\widetilde{f}^q\widetilde{x}_{i+1}^q \in (\widetilde{x}_1^q, \dots, \widetilde{x}_i^q, y_1, \dots, y_h)S$$

But $y_1, \dots, y_h, \widetilde{x}_1^q, \dots, \widetilde{x}_{i+1}^q$ is a permutable regular sequence in S, and so $(*)$ implies that

$$\widetilde{c}\widetilde{f}^q \in (\widetilde{x}_1^q, \dots, \widetilde{x}_i^q, y_1, \dots, y_h)S.$$

When we consider this modulo P, We have that $(y_1, \dots, y_h)$ is killed, and so

$$cf^q \in (x_1^q, \dots, x_i^q)$$

for all $q \geq N$, and this gives the desired conclusion. $\square$

Weak F-regularity: localization at maximal ideals and the Cohen-Macaulay property

We next want to prove that the property of being weakly F-regular is local on the maximal ideals of R. From this we will deduce that a weakly F-regular ring that is a homomorphic image of a Cohen-Macaulay ring is Cohen-Macaulay. We need two preliminary results.

Lemma. *Let R be any Noetherian ring, let M be a finitely generated R-module and $N \subseteq M$ a submodule. Then N is the intersection of a (usually infinite) family of submodules Q of M such that every M/Q is killed by a power of a maximal ideal of R.*

In particular, every ideal I of R is an intersection of ideals that are primary to a maximal ideal of R.

Proof. Let $u \in M - N$. Consider the family of submodules $M_1 \subseteq M$ such that $N \subseteq M$ and $u \notin M_1$. This family is nonempty, since it contains N. Therefore it has a maximal element Q. It will suffice to show that M/Q is killed by a power of a maximal ideal of R. Note that every nonzero submodule of M/Q contains the image of u, or else its inverse image in M will strictly contain Q but will not contain u.

We may replace M by M/Q and u by its image in M/Q. It therefore suffices to show that if $u \neq 0$ is in every nonzero submodule of M, then M is killed by a power of a maximal ideal, which is equivalent to the assertion that $\mathrm{Ass}\,(M)$ consists of a single maximal ideal. Let $P \in \mathrm{Ass}\,(M)$ and suppose that $P = \mathrm{Ann}_R v$, where $v \neq 0$ is in M. Then $Rv \cong R/P$, and every nonzero element has annihilator P. But $u \in Rv$, and so $P = \mathrm{Ann}_R u$. It follows that every associated prime of M is the same as $\mathrm{Ann}_R u$, and so there is only one associated prime. It remains to show that P is maximal. Suppose not, and consider $R/P \hookrightarrow M$. It will suffice to show that there is no element in all the nonzero ideals of R/P. Thus, it suffices to show that if $S = R/P$ is a Noetherian domain of dimension at least one, there is no nonzero element in all the nonzero ideals. This is true, in fact, even if we localize at a nonzero prime ideal m of S, for in S_m, there is no element in all of the ideals $m^n S_m$. $\square$

Proposition. *Let R be a Noetherian ring of prime characteristic $p > 0$, and let $\mathfrak{A}$ be an ideal of R.*

(a) *If $\theta : R \to S$ is such that S is flat Noetherian R-algebra and, in particular, if S is a localization of R, then $\theta(\mathfrak{A}_R^*) \subseteq (\mathfrak{A}S)_S^*$.*

(b) *Let m be a maximal ideal of R and suppose that $\mathfrak{A}$ is an m-primary ideal. Let $f \in R$. Then $f \in \mathfrak{A}^*_R$ if and only if $f/1 \in (\mathfrak{A}R_m)^*_{R_m}$.*

(c) *Under the hypotheses of part (b), $\mathfrak{A}$ is tightly closed in R if and only if $\mathfrak{A}R_m$ is tightly closed in R_m.*

Proof. (a) Let $f \in \mathfrak{A}^*_R$. The equation $cf^q \in \mathfrak{A}^{[q]}$ implies $\theta(c)\theta(f)^q \in (\mathfrak{A}S)^{[q]}$, and so we need only see that if $c \in R^\circ$ then $c \in S^\circ$. Suppose, to the contrary, that c is in a minimal prime $\mathfrak{q}$ of S. It suffices to see that the contraction $\mathfrak{p}$ of $\mathfrak{q}$ to R is minimal. But $R_\mathfrak{p} \to S_\mathfrak{q}$ is still faithfully flat, and the maximal ideal of $S_\mathfrak{q}$ is nilpotent, which implies that $\mathfrak{p}R_\mathfrak{p}$ is nilpotent, and so $\mathfrak{p}$ is minimal.

For part (b), we see from (a) that if $f \in \mathfrak{A}^*$ then $f \in (\mathfrak{A}R_m)^*$. We need to prove the converse. Suppose that $c_1 \in R_m^\circ$ has the property that $cf_1^q \in \mathfrak{A}^{[q]}R_m = (\mathfrak{A}R_m)^{[q]}$ for all $q \gg 0$. Then c_1 has the form c/w where $c \in R$ and $w \in R - m$. We may replace c_1 by wc_1, since w is a unit, and therefore assume that $c_1 = c/1$ is the image of $c \in R$. We next want to replace c by an element with the same image in R_m that is not in any minimal prime of R. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ be the minimal primes of R that are contained in m, so that the ideals $\mathfrak{p}_j R_m$ for $1 \leq j \leq k$ are *all* of the minimal primes of R_m. It follows that the image of $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k$ is nilpotent in R_m, and so we can choose an integer $N > 0$ such that $I = (\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k)^N$ has image 0 in R_m. If $c + I$ is contained in the union of the minimal primes of R, then by the coset form of prime avoidance, it follows that $cR + I \subseteq \mathfrak{p}$ for some minimal prime $\mathfrak{p}$ of R. Since $I \subseteq \mathfrak{p}$, we have that $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k \subseteq \mathfrak{p}$, and it follows that $\mathfrak{p}_j = \mathfrak{p}$ for some j, where $1 \leq j \leq k$. But then $c \in \mathfrak{p}_j$, a contradiction, since $c/1$ is not in any minimal prime of R°. Hence, we can choose $f \in I$ such that $c + f \in R^\circ$, and $c + f$ also maps to $c/1$ in R. We change notation and assume $c \in R^\circ$. Then $cf^q/1 \in \mathfrak{A}^{[q]}R_m$ for all $q \gg 0$. Since $\mathfrak{A}^{[q]}$ is primary to m, the ring $R/\mathfrak{A}^{[q]}$ has only one maximal ideal, $m/\mathfrak{A}^{[q]}$, and is already local. Hence,

$$R/\mathfrak{A}^{[q]} \cong (R/fA^{[q]})_m = R_m/\mathfrak{A}^{[q]}R_m.$$

It follows that $cf^q \in \mathfrak{A}^{[q]}$ for all $q \gg 0$, and so $f \in \mathfrak{A}^*_R$, as required.

Part (c) is immediate from part (b) and the observation above that $R_m/\mathfrak{A}R_m = R/\mathfrak{A}$, so that any element of $R_m/\mathfrak{A}R_m$ is represented by an element of R. □

Remark. Part (a) holds for any map $R \to S$ of Noetherian rings of prime characteristic $p > 0$ such that R° maps into S°. We have already seen another example, namely when $R \hookrightarrow S$ are domains.

Theorem. *The following conditions on R are equivalent.*

(1) *R is weakly F-regular.*

(2) *Every ideal of R primary to a maximal ideal of R is tightly closed.*

(3) *For every maximal ideal m of R, R_m is weakly F-regular.*

Proof. Statements (2) and (3) are equivalent by part (c) of the preceding Proposition, and (1) $\Rightarrow$ (2) is clear. Assume (2), and let I be any ideal of R. We need only show that I is tightly closed. If not, let $f \in I^* - I$. Since I is the intersection of the ideals containing I that are primary to maximal ideals, there is an ideal $\mathfrak{A}$ of R primary to a maximal ideal m such that $I \subseteq \mathfrak{A}$ and $f \notin \mathfrak{A}$. Since $\mathfrak{A}$ is tightly closed and $I \subseteq \mathfrak{A}$, we have $I^* \subseteq \mathfrak{A}$, and so $f \in \mathfrak{A}$, a contradiction. $\square$

Theorem. *Let R be a Noetherian ring of positive prime characteristic p that is a homomorphic image of a Cohen-Macaulay ring. If R is weakly F-regular, then R is Cohen-Macaulay.*

Proof. By the preceding result, it suffices to check this for $R_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R: the hypothesis of weak F-regularity is preserved. Thus, we may assume that R is local. We then know that R is normal, and so R is a domain, and it follows that the theorem on colon-capturing stated on the second page of the notes for this lecture holds. Hence, if $x_1, \ldots, x_d$ is a system of parameters for R, and $J_i = (x_1, \ldots, x_i)R$, $0 \leq i < d$, then $J_i :_R x_{i+1} \subseteq J_i^*$ and $J_i^* = J_i$. But the statement $J_i :_R x_{i+1} = J_i$ means that the image of x_{i+1} is not a zerodivisor in R/J_i, $0 \leq < d$, which means that $x_1, \ldots, x_d$ is a regular sequence on R. $\square$