Math 615, Winter 2020: Lecture of Friday, April 10

In this lecture we give brief discussions first of excellent rings, and then of F-rational rings in positive prime characteristic. The material in these fsubsections is primarily expository and is optional — it will not be needed for any quiz or problem set.

The condition of excellence is a convenient hypothesis: excellent rings have many of the same properties that finitely generated algebras over a field have.

F-rational rings are defined here as Noetherian rings R of positive prime characteristic p that are quotients of Cohen-Macaulay rings such that, for every local ring $R_\mathfrak{m}$ of the ring R at a maximal ideal $\mathfrak{m}$, the local ring is equidimensional (the quotient of $R_\mathfrak{m}$ by any minimal prime has the same dimension as $R_\mathfrak{m}$) and the ideal generated by one (equivalently, every) system of parameters is tightly closed. This condition implies that the ring is Cohen-Macaulay and normal, and turns out to be strictly weaker than F-regularity.

Excellent rings

Alexander Grothendieck introduced a class of Noetherian rings called *excellent* rings in his massive work, *Éléments de géométrie algébrique* IV, Publications Mathématiques de l'IHÉS **24** (1965), Section 7. These rings have many of the important properties shared by finitely generated algebras over a field (as mentioned above), or over the integers, or over a compete local ring. We give the definition and mention some basic properties, but we do not give a detailed treatment. However, in the sequel we will occasionally mention that a result about, for example, tight closure, generalizes to the excellent case.

One of the most important ideas underlying the usefulness of this notion is that for an excellent local ring R, there is an especially good way of showing that desirable properties of the completion $\widehat{R}$ are shared by R: this follows largely from the definition of a G-ring, discussed below. I feel that one of the most readable treatments of the theory of excellent rings is given in the book *Commutative Algebra* by H. Matsumura, Benjamin, New York, 1970.

A Noetherian K-algebra S is called *geometrically regular* over K (or $K \to S$ is called *geometrically regular*) if for every for every finite algebraic extension L of K, $L \otimes_K S$ is regular. This implies that S is regular, since it holds when $L = K$. If $L_0 \subseteq L$ are finite algebraic field extensions and $L \otimes_K S$ is regular, then $L_0 \otimes_K S$ is regular, because $L \otimes_K S$ is faithfully flat over $L_0 \otimes_K S$. Thus, the larger L is, the harder it is to satisfy the condition. However, if S is regular then $L \otimes_K R$ is regular whenever L is *separable finite algebraic* over K. Thus, if K has characteristic 0, or is perfect (and, of course, if K is algebraically closed), R is geometrically regular over K if and only if it is regular. If K has characteristic p, every finite algebraic extension is contained in a finite algebraic extension which consists of a finite purely inseparable extension followed by a finite separable extension. Hence,

$K \to R$ is geometrically regular if and only if for every purely inseparable finite algebraic extension L of K, we have that $L \otimes_K R$ is regular. Note that if R is an algebraic extension field of K, it is geometrically regular if and only if it is separable over K: otherwise, there will be nilpotents in $L \otimes KR$ when L is the splitting field of the minimal polynomial of an element of R that is not separable over K.

More generally, a map $R \to S$ of Noetherian rings is called *geometrically regular* if and only if it is flat with geometrically regular fibers. That is S is R-flat and for every prime P of R, with $\kappa_P = R_P/PR_P \cong \mathrm{frac}\,(R/P)$, the fiber $\kappa_P \otimes_R S$ is geometrically regular over κ_P.[1]

A Noetherian ring R is called a G-*ring* ("G" as in "Grothendieck") if for every local ring $A = R_P$ of R, the map $A \to \widehat{A}$ is geometrically regular.

This condition can fail even for a Noetherian discrete valuation domain V in characteristic $p > 0$. Let k be a perfect field of characteristic p and let $K = k(t_1, \ldots, t_n, \ldots)$ be the field generated by an infinite sequence of indeterminates over K. Let $K_0 = K^p$ and $K_n = K_0(t_1, \ldots, t_n)$, which will contain $t_1, \ldots, t_n$ but only t_h^p if $h > n$. Let $V_n = K_n[[x]] \subseteq K[[x]]$, which is a Noetherian discrete valuation domain with maximal ideal xV_n. Let $V = \cup_{n=0}^{\infty} V_n$, which is easily verified to be a Noetherian discrete valuation domain with maximal ideal xV. We still have $V \subseteq K[[x]]$, and it is easy to check that $\widehat{V}$ may be identified with $V[[x]]$. This means that the fraction field of $\widehat{V}$ is a purely inseparable the fraction field of V, and this extension is not geometrically regular.

We can finally give the definition of *excellence*. An *excellent* ring is a universally catenary Noetherian G-ring R such that in every finitely generated R-algebra S, the regular locus $\{P \in \mathrm{Spec}\,(S) : S_P \text{ is regular}\}$ is Zariski open. Therere are many ways to give this hypotheses with a superficially weaker assumption about the openness of the regular locus.

Excellent rings include the integers, fields, complete local rings, convergent power series rings, and are closed under taking quotients, localization, and formation of finitely generated algebras. The rings that come up in algebraic geometry, algebraic number theory, algebraic combinatorics, and several complex variables are excellent. Excellent rings tend very strongly to share the good behavior exhibited by rings that are finitely generated over a field. Here are some examples.

In connection with (4) of the list of properties that follows, note that the completion of an excellent local domain need *not* be a domain: if $S = K[x, y]$, where K is a field of characteristic different from 2 (there are similar examples in characteristic 2) $\mathfrak{m} = (x, y)S$ and $R = S_{\mathfrak{m}}/(y^2 - x^2(1 + x)$, then R is a domain (because $y^2 - x^2(1 + x)$ is irreducible, even over $K(x, y)$) but $\widehat{R}$ has two minimal primes (because $y^2 - x^2(1 + x)$ reduces in the completion $K[[x, y]]$: the point is that since $1 + x$ has a square root in $K[[x]]$, so does $x^2(1 + x)$.

[1]For those familiar with the notion of *smoothness*, when S is finitely presented as a R-algebra, S is smooth over R if and only if it is geometrically regular over R. There is a complete treatment in the Lecture Notes from Math 615, Winter 2017.

For a detailed treatment of the items (5) and (6), we refer to M. Hochster and C. Huneke, *F-regularity, test elements, and smooth base change*, Trans. Amer. Math. Soc. **346** (1994), 1–62.

Recall that a local ring R is *equidimensional* if for every minimal prime $\mathfrak{p}$ of R, the Krull dimension of $R/\mathfrak{p}$ is the same as the Krull dimension of R.

(1) The normalization of an excellent domain R is a finitely generated R-module.

(2) The completion of a reduced excellent local ring is reduced.

(3) The completion of a normal excellent local domain is normal.

(4) The completion of any excellent reduced, equidimensional local ring R is reduced and equidimensional. In particular, the completion of an excellent local domain is reduced and equidimensional.

(5) (Colon-capturing in excellent local rings.) If R is an excellent equidimensional ring of positive prime characteristic, $x_1, \ldots, x_d$ is a system of parameters, and $J_i := (x_1, \ldots, x_i)R$, $J_i :_R x_{i+1} \subseteq J_i^*$.

(6) (Existence of test elements.) Let R be a localization of a reduced finitely generated algebra over an excellent semilocal ring of positive prime characteristic. Let $c \in R$ be any element not in any minimal prime of R such that R_c is regular (such elements exist). Then c has a power that is a test element for tight closure in R.

F-rational rings

We have defined a Noetherian ring of positive prime characteristic p to be F-*rational* if it is a homomorphic image of a Cohen-Macaulay ring R and for the localization at each maximal ideal, the local ring is equidimensional and one (equivalently, every) system of parameters is tightly closed. For a detailed treatment see M. Hochster and C. Huneke, *F-regularity, test elements, and smooth base change*, Trans. Amer. Math. Soc. **346** (1994), 1–62, especially Theorem 4.2. One can show a local ring $(R, \mathfrak{m})$ satisfying this condition is a normal domain. In the domain case, the problem of showing that if one system of parameters is tightly closed then all systems of iis tightly closed, is addressed in Problem Set 5. Once one knows this it follows that if $x_1, \ldots, x_d$ is a system of parameters, then all of the ideals $J_i = (vectxi)R$, $0 \leq i \leq d$ are tightly closed since

$$J_i = \bigcap_{t=1}^{\infty} (x_1, \ldots, x_i, x_{i+1}^t, \ldots, x_d^t)R$$

(since J_i is closed in the $\mathfrak{m}$-adic topology on R). Thus, in an F-rational local ring, every ideal generated by part of a system of parameters is tightly closed. The colon-capturing theorem for quotients of Cohen-Macaulay rings may be applied, and then exactly the same proof as used in the Theorem at the end of the preceding lecture shows that R is Cohen-Macaulay. Thus, every F-rational ring is Cohen-Macaulay as well as normal.

This paragraph assumes additional background in algebraic geometry. A finitely generated algebra over a field K of characteristic 0 is said to have *rational singularities* if is Cohen-Macaulay, normal, and the higher direct images of the structure sheaf of a desingularization are 0. (This characterization is redundant, and there are other characterizations.) This notion is independent of base change of the field. In considering an algebra like this over a field K, K may be replaced by a subfield that is finitely generated over $\mathbb{Q}$ and contains the coefficients of the defining equations of the radical ideal that is used to define the algebra as a quotient of a polynomial ring. In fact, one can choose a finitely generated $\mathbb{Z}$-algebra $A \subseteq K$ and a finitely generated A-algebra R_A such that the coordinate ring of the original algebraic set is $R = K \otimes_A R_A$. By a theorem of Karen Smith and Nobuo Hara (cf. Karen E. Smith, *F-rational rings have rational singularities*, Amer. J. Math. **119** (1997), 159–180 and Nobuo Hara, *A characterization of rational singularities in terms of injectivity of Frobenius maps*, Amer. J. Math. **120** (1998) 981–996 R has rational singularities if and only if for all maximal ideals μ in Zariski open dense subset of the maximal spectrum of A, with $\kappa := A/\mu$ (note that this is a *finite* field) one has that $R_\kappa := \kappa \otimes_A R_A$ is F-rational. Thus, notions of "good behavior" defined in terms of tight closure and other positive characteristic phenomena can be used to characterize related notions of "good behavior" over fields of characteristic 0. There are many other examples, and there is a considerable literature on the relationship between characteristic p properties and properties defined over fields of characteristic 0.