Math 615, Winter 2020: Lecture of Monday, April 13

In this lecture we give a summary of the structure theory of complete local rings. The statements of some of the results (these will be clearly indicated) are required material. Complete proofs of the results stated in these notes will be provided in a Supplement, which will be both posted and distributed by e-mail. But going through the proofs of the results is optional.

Summary of the structure theory of complete local rings over a field

We begin with some of the structure theory of complete local rings in the case where the ring contains a field. One key point is that if $(R, \mathfrak{m}, K)$ is a complete local ring that contains a field, then R contains a field $\widetilde{K}$ such that the canonical surjection $R \twoheadrightarrow R/\mathfrak{m} = K$ carries $\widetilde{K}$ isomorphically onto K. $\widetilde{K}$ is called a *coefficient field* for R. This means that if a complete local ring contains any field, it contains an isomorphic copy of its residue class field. If R contains a field of characteristic 0, one may take any maximal subfield of R (such exist by Zorn's lemma) as the coefficient field. The proof of the result is much more difficult in characteristic $p > 0$. However, if K is perfect, the choice of $\widetilde{K}$ is unique: it must be $\bigcap_e F^e(R)$, the set of all elements of R that have p^e th roots in R for all $e \geq 1$. This does turn out to be a field: it is disjoint from $\mathfrak{m}$, since any element in it that is in $\mathfrak{m}$ must be in $\mathfrak{m}^{[p^e]}$ for all e and, hence, 0.

In equal characteristic 0 and other positive characteristic cases, the choice of $\widetilde{K}$ is usually not unique. However, it is typical to use the same letter K for both a choice of $\widetilde{K}$, i.e., a choice of coefficient field, as for the residue class field, and we usually do so in the sequel.

Two very important points in the structure theory: if $K \hookrightarrow R$ is a coefficient field for the complete local ring $(R, \mathfrak{m}, K)$, and $x_1, \ldots, x_n \in \mathfrak{m}$, there is unique continuous (with respect to the respective maximal ideal-adic topologies) K-algebra map from the formal power series ring $\theta : K[[X_1, \ldots, X_n]] \to R$ such that $X_i \mapsto x_i$, $1 \leq i \leq n$. If $x_1, \ldots, x_n$ generate $\mathfrak{m}$, this map is surjective. If $n = d = \dim(R)$ and $x_1, \ldots, x_d$ is a system of parameters for R, the map is injective, and R is module-finite over the image. Thus, if R is regular, $n = d$, and $x_1, \ldots, x_d$ is a minimal set of generators of $\mathfrak{m}$ (i.e., a regular system of parameters), then $\theta : K[[X_1, \ldots, X_d] \to R$ is an isomorphism. Thus, every complete local ring containing a field is a homomorphic image of a formal power series ring, and is also module-finite over a subring (necessarily of the same dimension) which is a formal power series ring. Both of these statements are analogues of statements that are true for finitely generated algebras over a field and polynomial rings. The second statement is analogous to Noether normalization. It also follows that a complete local ring containing a field is regular iff it is isomorphic with a formal power series ring $K[[x_1, \ldots, x_d]]$.

There is a very satisfactory structure theory for complete local rings that do not necessarily contain a field. These have what is called *mixed characteristic*: the residue class field has characteristic $p > 0$, while the ring itself has characteristic 0 or p^h for some $h \geq 2$. Complete Noetherian discrete valuation domains V in which a prime integer p generates the maximal ideal (like the p-adic integers) can be used as coefficient rings in the domain case (there cannot be a coefficient field, but the residue field is the residue field of one of these coefficient rings). In other cases, there may be a coefficient ring of the form V/p^hV. It is still true in mixed characteristic that every complete local ring is a homomorphic image of a formal power series ring $V[[X_1, \ldots, X_n]]$ with V as above. Every complete local domain R of mixed characteristic is module-finite over a subring of the form $V[[X_2, \ldots, X_d]$ where the images of $p, X_2, \ldots, X_d$ form a system of parameters in R. Moreover, every mixed characteristic complete regular local ring has either the form $V[[X_2, \ldots, X_d]$ (if p is not in the square of the maximal ideal) or else has the form $V[[X_2, \ldots, X_d, X_{d+1}]/(f)$, where $f = p - g$ and g is in the square of the maximal ideal of the formal power series ring. There is a complete treatment in the Supplement, but we won't use the mixed characteristic theory in this course.

We shall use the structure theory of complete local regular local rings in equal characteristic p to show that in prime characteristic $p > 0$, the Frobenius endomorphism and its iterations, i.e., its powers under compostion, are flat. It is then immediate that they are faithfully flat, since the extension of the maximal ideal $\mathfrak{m}$ underF^e is $\mathfrak{m}^{[p^e]}$, which is contained in $\mathfrak{m}$ and so a proper ideal. This will enable us to prove that all regular rings are F-regular in characteristic $p > 0$.

The Discussion that follows and the *statements* of the three Theorems on the next page are required material. The full proofs are given in the Supplement and are not required.

Discussion. Let $(R, \mathfrak{m}, . K)$ be a complete local ring that is an A-algebra for some ring A and let $x_1, \ldots, x_n \in \mathfrak{m}$. We have a unique A-homomorphism θ_0 of the polynomial ring $A[X_1, \ldots, X_n] \to R$ such that $X_i \mapsto x_i$, $1 \leq i \leq n$. This extends uniquely to the power series ring $B := A[[X_1, \ldots, X_n]]$ if we require continuity with respect to the $(X_1, \ldots, X_n)B$-adic and fm-adic topologies. To see this, consider an element $u \in B$ and let u_h be the sum of all terms of u involving only monomials in the power series for u of degree at most h. Then u is the limit of the u_n, which are polynomials, in the $(X_1, \ldots, X_n)B$-adic topology. Note that $\theta_0(u_{h+1}) - \theta_0(u_h)$ is an A-linear combination of monomials of degree $h + 1$ in the elements $x_i = \theta(X_i)$, and so is in $\mathfrak{m}^{h+1}$. Hence, the elements $\theta_0(u_h)$ form a Cauchy sequence in R, and have a unique limit in R, which we denote $\theta(u)$. Clearly, the continuity assumption forces the homomorphism extending θ_0 to be θ. It is easy to check that θ preserves addition and multiplication, that it is A-linear, and that it extends θ_0. This proves the existence and uniqueness of θ. In fact, if $\underline{i} := (i_1, \ldots, i_n)$ varies in $\mathbb{N}^n$, then

$$\theta : \sum_{\underline{i}} a_{\underline{i}} X^{i_1} \cdots X_n^{i_n} \mapsto \sum_{\underline{i}} a_{\underline{i}} x_1^{i_1} \cdots x_n^{i_n}.$$

We can now state:

Theorem. *Let notation be as in the preceding discussion.*

(a) If $A \to R \to K$ is a surjection and $x_1, \ldots, x_n$ generate $\mathfrak{m}$, then $\theta : A[[X_1, \ldots, X_n]] \to R$ is surjective.

(b) Let $K \hookrightarrow R$ be a coefficent field for R, so that $K \hookrightarrow R \twoheadrightarrow K$ is an isomorphism. Suppose that $x_1, \ldots, x_d$ is a system of parameters for R and that $n = d$. Then $\theta : K[[X_1, \ldots, X_n]] \to R$ is injective, and R is module-finite over the image, which is isomorphic to a formal power series ring.

(c) Suppose that R is regular, that $K \hookrightarrow R$ is a coefficient field, and the $x_1, \ldots, x_n$ is a regular system of parameters, i.e., a system of parameters that generates $\mathfrak{m}$. Then $\theta : K[[X_1, \ldots, X_n]]$ is an isomorphism.

For the proof, we refer to the Supplement on complete local rings, but the *statement* of this result is required material. We note that in part (b), the image of the map θ is often denoted $K[[x_1, \ldots, x_d]]$.

Since an Artin local ring that contains a field is automatically complete, we have:

Corollary. *An Artin local ring R that contains a field contains a coefficient field K, and so is a finite-dimensional vector space over K.*

The final statement in the Corollary just above is immediate from the fact that the length of the Artin local ring is finite and, when there is a coefficient field, length coincides with vector space dimension over the coefficient field.

The statement of the following very important result is required:

Theorem. *Every complete local ring that contains a field has a coefficient field.*

The proof is given in the Supplement.

Given the preceding two results, we immediately have the following result, whose statement is required material:

Theorem. *Let R be a complete local ring that contains a field.*

(a) R is a homomorphic image of a formal power series ring $K[[X_1, \ldots, X_n]]$.

(b) (Formal Noether normalization.) If $x_1, \ldots, x_d$ is a system of parameters for R, then R is module finite over the formal power series subring $K[[x_1, \ldots, x_d]]$.

(c) If R is regular local and $x_1, \ldots, x_d$ is a regular system of parameters, then R is isomorphic with the formal power series ring $K[[x_1, \ldots, x_d]]$. $\square$

Remark. The structure theorems show that formal power series rings over K have a great many K-automorphisms. We illustrate this point with one example. Consider the formal power series ring $K[[x,y,z]]$, with $\mathfrak{m} = (x,y,z)$. The elements $u := x + y^2 + z^3$, $v := y + x^7 + xz^{11}$, and $w := z + x^{13} + y^{19} + x^{67}y^{23}z^{101}$ are a regular system of parameters, since they generate $\mathfrak{m}/\mathfrak{m}^2$. Thus, $K[[u,v,w]] \subseteq K[[x,y,z]]$ is actually an equality, and x, y, and z can all be expressed as power series in u, v, and w. Moreover, there is an $\mathfrak{m}$-adically continuous K-automorphism of this formal power series ring that maps x, y, and z to u, v, and w, respectively. The higher degree terms in the expressions for u, v, and w were chosen more or less randomly.