Math 615, Winter 2020: Lecture of Wednesday, April 15

In this Lecture we first give two proofs of the (faithful) flatness of the Frobenius endomorphism (and, hence, of its iterates) for regular rings of positive prime characteristic p. Both reduce to the local case. The first then reduces to the complete case and utilizes the structure theory of complete regular local rings over a field. The second makes use of Problem **6.** in Problem Set #4, but ultimately depends on the homological characterization of regular local rings and understanding of Tor. The flatness of Frobenius is then used to prove that every ideal is tightly closed in a regular ring of positive prime characteristic p: in fact every submodule of every finitely generated module is tightly closed.

The remainder of the material is optional. First, the important that if every ideal is tightly closed, then every submodule of every module is tightly closed is proved *in complete generality*. The treatment uses one result in the literature that is not established in the course. Along the way, there is a discussion of Gorenstein local rings and of approximately Gorenstein local rings. It is shown that a Gorenstein Artin local ring is injective as a module over itself in the case where the ring contains a field. This is true more generally. A final section gives a treatment of tight closure for modules using the Frobenius functors introduced in Problem **5.** of the Problem Set #5. This approach makes it clear that the definition we gave earlier for tight closure of N in M is independent of the choices we made (such as the free module that is mapped onto M).

The flatness of the Frobenius endomorphism for all regular rings of positive prime characteristic

We next want to establish the assertion made earlier that the Frobenius endomorphism is flat for every regular Noetherian ring of prime characteristic $p > 0$. Faithful flatness is then obvious. We give two proofs of this: the first relies on the structure theory of complete local rings. In both proofs, we want to reduce to the case where the ring is local. In the first proof we then reduce to the case where the ring is complete local. We first observe the following:

Proposition. *Let $\theta : (R, m, K) \to (S, \mathfrak{n}, L)$ be a homomorphism of local rings that is* local, *i.e., $\theta(m) \subseteq \mathfrak{n}$. Then S is flat over R if and only if for every injective map $N \hookrightarrow M$ of finite length R-modules, $S \otimes_R N \hookrightarrow S \otimes_R M$ is injective.*

Proof. The condition is obviously necessary. We shall show that it is sufficient. Since tensor commutes with direct limits and every injection $N \hookrightarrow M$ is a direct limit of injections of finitely generated R-modules, it suffices to consider the case where $N \subseteq M$ are finitely generated. Suppose that some $u \in S \otimes_R N$ is such that $u \mapsto 0$ in $S \otimes_R M$. It will suffice to show that there is also such an example in which M and N have finite length. Fix any integer $t > 0$. Then we have an injection

$$N/(m^t M \cap N) \hookrightarrow M/m^t M$$

and there is a commutative diagram

$$\begin{array}{ccc} S \otimes_R N & \xrightarrow{\iota} & S \otimes_R M \\ f\big\downarrow & & g\big\downarrow \\ S \otimes_R \big(N/(m^t M \cap N)\big) & \xrightarrow{\iota'} & S \otimes_R \big(M/m^t M\big) \end{array}.$$

The image $f(u)$ of u in $S \otimes_R \big(\big(N/(m^t M \cap N)\big)$ maps to 0 under ι', by the commutativity of the diagram. Therefore, we have the required example provided that $f(u) \neq 0$. However, for all $h > 0$, we have from the Artin-Rees Lemma that for every sufficiently large integer t, $m^t M \cap N \subseteq m^h N$. Hence, the proof will be complete provided that we can show that the image of u is nonzero in

$$S \otimes_R (N/m^h N) \cong S \otimes_R \big((R/m^h) \otimes_R N\big) \cong (R/m^h) \otimes_R (S \otimes_R N) \cong (S \otimes_R N)/m^h (S \otimes_R N).$$

But

$$m^h (S \otimes_R N) \subseteq \mathfrak{n}^h (S \otimes_R N),$$

and the result follows from the fact that the finitely generated S-module $S \otimes_R N$ is $\mathfrak{n}$-adically separated. □

Lemma. *Let $(R, m, K) \to (S, \mathfrak{n}, L)$ be a local homomorphism of local rings. Then S is flat over R if and only if $\widehat{S}$ is flat over $\widehat{R}$, and this holds iff $\widehat{S}$ is flat over R.*

Proof. If S is flat over R then, since $\widehat{S}$ is flat over S, we have that $\widehat{S}$ is flat over R. Conversely, if $\widehat{S}$ is flat over R, then S is flat over R because $\widehat{S}$ is faithfully flat over S: if $N \subseteq M$ is flat but $S \otimes_R N \to S \otimes_R M$ has a nonzero kernel, the kernel remains nonzero when we apply $\widehat{S} \otimes_S \_$, and this has the same effect as applying $\widehat{S} \otimes_R \_$ to $N \subseteq M$, a contradiction.

We have shown that $R \to S$ is flat if and only $R \to \widehat{S}$ is flat. If $\widehat{R} \to \widehat{S}$ is flat then since $R \to \widehat{R}$ is flat, we have that $R \to \widehat{S}$ is flat, and we are done. It remains only to show that if $R \to S$ is flat, then $\widehat{R} \to \widehat{S}$ is flat. By the Proposition, it suffices to show that if $N \subseteq M$ have finite length, then $\widehat{S} \otimes N \to \widehat{S} \otimes M$ is injective. Suppose that both modules are killed by m^t. Since $S/m^t S$ is flat over R/m^t, if Q is either M or N we have that

$$\widehat{S} \otimes_{\widehat{R}} Q \cong \widehat{S}/m^t \widehat{S} \otimes_{\widehat{R}/m^t \widehat{R}} Q \cong \widehat{S}/m^t \widehat{S} \otimes_{R/m^t} Q \cong \widehat{S} \otimes_R Q,$$

and the result now follow because $\widehat{S}$ is flat over R. □

We are now ready to prove:

Theorem. *Let R be a regular Noetherian ring of prime characteristic $p > 0$. Then the Frobenius endomorphism $F : R \to R$ is flat.*

Proof. To distinguish the two copies of R, we let S denote the right hand copy, so that $F : R \to S$. The issue of flatness is local on R, and if P is prime, then $(R - P)^{-1}S$ is the localization of S at the unique prime Q lying over P (if we remember that S is R, then Q is P), since the p th power of every element of $S - Q$ is in the image of $R - P$. Hence, there is no loss of generality in replacing R by R_P, and we henceforth assume that (R, m, K) is local.

By the preceding Lemma, $F : R \to R$ is flat if and only if the induced map $\widehat{R} \to \widehat{R}$ is flat, and this map is easily checked to be the Frobenius endomorphism on $\widehat{R}$. We have now reduced to the case where R is a complete regular local ring. By the structure theory for complete regular local rings, we have $R = K[[x_1, \ldots, x_n]]$ where K is a field of characteristic p. By the final Theorem of the Lecture Notes of March 13, $F : K[x_1, \ldots, x_n] \to K[x_1, \ldots, x_n]$ makes $K[x_1, \ldots, x_n]$ into a free algebra over itself. It follows that it is flat over itself, and this remains true when we localize at $(x_1, \ldots, x_n)$. By the preceding Lemma, we still have flatness after we complete both rings. Completing yields

$$F : K[[x_1, \ldots, x_n]] \to K[[x_1, \ldots, x_n]],$$

which proves the flatness result we need. $\square$

Second proof of the flatness of Frobenius in regular rings This argument is very short, but depends both on the homological characterization of regular local rings and the use of the functor Tor. Consider $F : R \to R$. By Problem **6.** of Problem Set #4, it suffices to show that given a system of parameters $x_1, \ldots, x_d$ for R, its image under F is a regular sequence is the target copy of R. This is obvious, since the image $x_1^p, \ldots, x_d^p$ is a system of parameters for the target copy of R. $\square$

Remark. It is also true that a Noetherian ring of positive prime characteristic p is regular if and only if the Frobenius endomorphism is flat. Cf. E. Kunz, *Characterizations of regular lcoal rings of characteristic p*, Amer. J. Math. **91** (1969) 772–784.

Regular rings are weakly F-regular: the general case

We can now give the application of this result that we have been intending for some time. We first generalize a previous lemma about flatness and its application to the Frobenius endomorphism over regular rings.

Lemma. *Let $R \to S$ be a flat ring homomorphism, let $H \subseteq G$ be R-modules, and let N be a finitely generated submodule of G. Identify $S \otimes_R H$ and $S \otimes_R N$ with submodules of $S \otimes_R G$. Then $(S \otimes_R H) :_S (S \otimes_R N) = S \otimes (H :_R N)$.*

Proof. Let $u_1, \ldots, u_n$ generate N and consider the composite map $R \to G^{\oplus n} \twoheadrightarrow (G/H)^{\oplus n}$ such that the map on the left sends $r \mapsto (ru_1, \ldots, ru_n)$ and the map on the right is the direct sum of n copies of the quotient surjection $G \twoheadrightarrow G/H$. The kernel of this map is $H : RN$, so that $0 \to H :_R N \to R \to (G/H)^{\oplus n}$ is exact. When we apply $S \otimes_R \_$, the fact that S is R-flat implies that

$$0 \to S \otimes_R (H :_R N) \to S \to \big((S \otimes G)/(S \otimes H)\big)^{\oplus n}$$

is exact. The required result follows because the kernel of the map on the right is also $(S \otimes_R H) :_S (S \otimes_R N)$. □

If we apply this when $S = R$ is regular, $R \to R$ is the e th iteration of the Frobenius endomorphism, G is free, and N is generated by one element u, we obtain:

Corollary. *If R is regular of characteristic $p > 0$, G is free, $H \subseteq G$, and $u \in G$, then for all $q = p^e$, $e \in \mathbb{N}$, we have that $H^{[q]} :_R u^q = (H :_R u)^{[q]}$.* □

Theorem. *Let R be a regular Noetherian ring of prime characteristic $p > 0$. Then every ideal I of R is tightly closed. In fact, every submodule of every finitely generated module is tightly closed.*

Proof. It suffices to prove the second assertion, and by it suffices to prove that every submodule H of every finitely generated free R-module G is tightly closed.

Suppose $u \in H^* - H$ in G and $c \in R$ not in any minimal prime and satisfies $cu^q \in H^{[q]}$ for all $q \gg 0$. We may replace R by its localization at a maximal ideal $\mathfrak{m}$ in the support of $(I + Ru)/I$, G by $G_\mathfrak{m}$, H by $H_\mathfrak{m} \subseteq G_\mathfrak{m}$ and u by its image in the local ring $R_\mathfrak{m}$. The image of c in $R_\mathfrak{m}$ is still not in any minimal prime, i.e., it is not 0. Hence, we still have that $u \in H^*_\mathfrak{m} - H_\mathfrak{m}$ in $G_\mathfrak{m}$. Thus, we may assume without loss of generality that R is local. Then for some q_0,

$$c \in \bigcap_{q \geq q_0} H^{[q]} :_R u^q = \bigcap_{q \geq q_0} (H :_R u)^{[q]} \subseteq \bigcap_{q \geq q_0} m^{[q]} \subseteq \bigcap_{q \geq q_0} m^q = (0),$$

a contradiction. The leftmost equality in the display follows from the Corollary above. □

If every ideal is tightly closed, then every submodule of every finitely generated module is tightly closed

The rest of this lecture consists entirely of optional material, first a section devoted to the proof of the characteristic $p > 0$ fact stated just above. Thus, either condition may be used to define weakly F-regular rings. The material is self-contained, except for one fact not proved in this course. A reference is provided. The "external" statement that we need is shown in boldface in the first full paragraph of the page after the next.

The last part of the lecture, also optional, gives a treatment of tight closure for modules using Frobenius functors.

Discussion. It is always true that if a Noetherian ring of positive prime characteristic is weakly F-regular in the sense that every ideal is tightly closed, then every submodule of every finitely generated module is tightly closed. In this discussion we explain why, but part of the argument relies on a reference to material outside the course. We also need a fact about zero-dimensional Gorenstein rings, but a self-contained treatment for the case where the ring contains a field is provided.

Beginning of the argument. Suppose that every ideal of R is tightly closed. Suppose $N \subseteq M$ are finitely generated modules such that $u \in M$ is in the tight closure of N but not in N. Then this remains true when we localize at a suitable maximal ideal of R, one that contains the annihilator of the class of u in M/N. Hence, we may assume without loss of generality that $(R,\, m,\, K)$ is local. Second, we may replace N by a maximal submodule N' of M such that $N \subseteq N'$ and $u \notin N'$. We replace M by M/N' and u by its image in M/N'. Thus, we may assume that u is in every nonzero submodule of M and is in the tight closure of 0 but not 0. This implies that the only associated prime of M is $\mathfrak{m}$: if $P \neq 0\mathfrak{m}$ were associated, so $R/P \hookrightarrow M$ and u would in (the image of) all the powers of $\mathfrak{m}/P$, and the intersection of those powers is 0. It follows that M has finite length, and we can choose n such that $\mathfrak{m}^n$ kills M.]. Moreover, since u is in every nonzero submodule of M, it must be, up to a unit multiplier, the unique nonzero element in the socle of M (if the socle had dimenison two or more as K-vector space, u could not be a multiple of each of two linearly independent elements in the socle).

Digression: Gorenstein and approximately Gorenstein rings. Because every ideal of the local ring R is tightly closed, we know that R is normal. The paper [M. Hochster, *Cyclic purity versus purity in excellent Noetherian rings*, Trans. Amer. Math. Soc. **231** (1977) 463–488] introduces and studies the notion of *approximately Gorenstein rings*. A local ring $(R,\, m,\, K)$ is called *Gorenstein* if it is Cohen-Macaulay of type 1. (It is not obvious but true that this property passes to localizations of the ring at prime ideals: see the notes on Local Cohomology that have been added to the Web page for the course.) A Noetherian ring is then define to be Gorenstein if its localizations at all prime ideals are Gorenstein). The type condition means that if $x_1,\, \ldots\,, x_d$ is any system of parameters, where d is the dimenson of R, then the Artin ring $R/(x_1,\, \ldots\,, x_d)R$ has a one-dimensional socle, and this is equivalent to the condition that the socle, which is the annihilator of $\mathfrak{m}$ in $R/(x_1,\, \ldots\,, x_d)R$, is isomorphic to one copy of $R/\mathfrak{m}$. Under this assumption it is easy to see that the socle is contained in every nonzero submodule, because every nonzero submodule has at least one element killed by m.[1] In general, the condition for a local ring of dimenson 0 (i.e., an Artin local ring) to be Gorenstein is that the socle be one-dimensional: in dimension 0, the Cohen-Macaulay assumption is automatic.

A local ring $(R,\, m,\, K)$ is called *approximately Gorenstein* if for every power $\mathfrak{m}^n$ of the maximal ideal, there is an $\mathfrak{m}$-primary ideal I such that $I \subseteq \mathfrak{m}^n$ and R/I is Gorenstein.

[1] If the submodule is N take s maximum such that $\mathfrak{m}^s N \neq 0$, and then every element of $\mathfrak{m}^s N$ is in the socle. Here, s may be 0, in which case N itself is killed by $\mathfrak{m}$.

Note that an $\mathfrak{m}$-primary ideal I_n has the property that R/I_n is Gorenstein if and only if I is *irreducible*, i.e., not the intersection of two strictly larger ideals. Every Gorenstein ring R is approximately Gorenstein: if R is 0-dimensional, we can take all the I_n to be 0, while if the dimension of R is $d > 0$ and $x_1, \ldots, x_d$ is a system of parameters for R, we may take $I_n = (x_1^n, \ldots, x_d^n)R$. It is clear that R/I_n is Gorenstein, since $x_1^n, \ldots, x_d^n$ is a system of parameters for R.

It turns out that being approximately Gorenstein is not a strong condition on R. In the paper referenced above, the condition is characterized, and it is shown that **every normal local ring is approximately Gorenstein**, which is what we need here.[2]

The other fact that we need is that *a 0-dimensional Gorenstein local ring is an injective module over itself.*[3] This is a very important result and is proved, for example, in the addition to the Web page on Local Cohomology, which has been posted. We give a self-contained proof here for the case where the ring contains a field (assuming the existence of coefficient fields from the structure theory), which is the only case we need.

Proof that a 0-dimensional Gorenstein local ring that contains a field is injective as a module over itself. Let (R, m, K) be a 0-dimensional local ring with a one-dimensional socle. Suppose that R contains a field. Then R has a coefficient field $K \subseteq R$: we fix such a coefficient field. We shall show that $E = \mathrm{Hom}_K(R, K)$ is an injective R-module (the R-module structure on the first input gives an R-module structure on E) and that it is isomorphic with R. This proves that R is injective as an R-module. The fact that E is injective follows from the following isomorphism of functors: $M \to \mathrm{Hom}_R(M, E)$ and $M \to \mathrm{Hom}_K(M \otimes_R, K)$. The latter is clearly exact, and the isomorphism is a consequence of the adjointness of tensor and Hom: $\mathrm{Hom}_K(M \otimes_R, K) \cong \mathrm{Hom}_R\big(M, \mathrm{Hom}_K(R, K)\big)$. To show that $R \cong \mathrm{Hom}_K(R, K)$ as an R-module, first note that since length is the same as K-vector space dimension here, they have the same length. To complete the proof, let $K \cong Ku = Ru$ be the socle in R. Choose a K-linear map $\theta : R \to K$ such that $u \mapsto 1$. We shall show that θ is not killed by any nonzero r in R. It follows that $R \cong R\theta \subseteq E$. But then since $R\theta$ has the same length as E, they are equal. To see that $r\theta \neq 0$, note that rR must meet Ku, and it follows that there exists $a \in R$ such that $ar = u$. Then $r\theta(a) = \theta(ar) = \theta(u) = 1$, so $r\theta$ is not 0. $\square$

Conclusion of the argument. We now come back to the situation at the end of the paragraph labeled **Beginning of the argument.** As already observed we know that R is normal, and so R is approximately Gorenstein. Thus, R has an irreducible $\mathfrak{m}$-primary ideal I such that $I \subseteq \mathfrak{m}^n$, where $\mathfrak{m}^n$ kills M. Hence, M is a module over $A := R/I$, which is a 0-dimensional Gorenstein ring. Consider the map from $Ru \cong K = R/\mathfrak{m}$ to R/I that identifies u with a generator of the socle in A, which is one copy of $R/\mathfrak{m}$. Since A is injective as an A-module, this extends to an A-linear map $\alpha : M \to A$. This map must be injective: if the kernel were nonzero it would contain u, and u does not map to 0. Thus,

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[2] It is also shown that every local ring that has depth at least two on its maximal ideal is approximately Gorenstein, and that every reduced excellent local ring is approximately Gorenstein.

[3] In fact, a local ring is Gorenstein if and only if it has finite injective dimension as a module over itself. In this case, the Cohen-Macaulay property follows. Moreover, the injective dimension is always the same as the Krull dimension. Again, this is proved in the addition to the Web page on Local Cohomology.

$M \hookrightarrow R/I$, and this is also an embedding as an R-modules. Since u is in the tight closure of 0 on M, its image v in R/I is in the tight closure of 0 in R/I. Suppose v is the image of w in R. Then w is in the tight closure of I in R, but not in I, a contradiction. $\square$

A functorial treatment of tight closure for modules

The material in this section is optional, although it may be helpful to read the more detailed treatment here of the Frobenius functors (also called *Peskine-Szpiro functors*) that are introduced in Problem **5.** of the Problem Set #5. The new treatment makes it clear that whether an element of M is in the tight closure of a submodule $N \subseteq M$ is independent of the choices (such as a free module mapping onto M) in our original definition. The fact that this definition is equivalent to the earlier one is proved at the end of the lecture.

Remark. Let S be Noetherian of positive prime characteristic p. It is important to note that if R is a homomorphic of S, and $N \subseteq M$ are Noetherian R modules, then the tight closure of N in M **depends very heavily** on whether one is worknig over R or S. This is already true in the case of ideals. E.g., let $S = K[[x, y]]$ and $R = S/(x^3 - y^2)S \cong K[[t^2, t^3]]$. Working over S, the submodule xR of R is tightly closed, since S is regular. But working over R, the tight closure of xR is $(x, y)R$ (note that $K[[t]]$ is a module-finite extension of R, and the image of y is $t^3 \in xK[[t]] \cap R = t^2K[[t]] \cap K[[t^2, t^3]]$.

Much more about tight closure may be found in the addition to the Web page entitled *Foundations of Tight Closure Theory*, which are lecture notes from a course on tight closure.

If R is a ring, we use the notation R° to denote the set of elements in R not in any minimal prime. These are the elements that are available to be the constant element of R used as a multiplier in the definition of tight closure.

In this treatment of tight closure for modules we use the Frobenius functors, which we view as special cases of base change. We first review some basic facts about base change.

Base change. If $f : R \to S$ is an ring homomorphism, there is a base change functor $S \otimes_R \_$ from R-modules to S-modules. It takes the R-module M to the R-module $S \otimes_R M$ and the map $h : M \to N$ to the unique S-linear map $S \otimes_R M \to S \otimes_R N$ that sends $s \otimes u \mapsto s \otimes h(u)$ for all $s \in S$ and $u \in M$. This map may be denoted $\mathrm{id}_S \otimes_R h$ or $S \otimes_R h$. Evidently, base change from R to S is a covariant functor. We shall temporarily denote this functor as $\mathcal{B}_{R \to S}$. It also has the following properties.

(1) Base change takes R to S.

(2) Base change commutes with arbitrary direct sums and with arbitrary direct limits.

(3) Base change takes R^n to S^n and free modules to free modules.

(4) Base change takes projective R-modules to projective S-modules.

(5) Base change takes flat R-modules to flat S-modules.

(6) Base change is right exact: if

$$M' \to M \to M'' \to 0$$

is exact, then so is

$$S \otimes_R M' \to S \otimes_R M \to S \otimes_R M'' \to 0.$$

(7) Base change takes finitely generated modules to finitely generated modules: the number of generators does not increase.

(8) Base change takes the cokernel of the matrix $\big(r_{ij}\big)$ to the cokernel of the matrix $\big(f(r_{ij})\big)$.

(9) Base change takes R/I to S/IS.

(10) For every R-module M there is a natural R-lineaar map $M \to S \otimes M$ that sends $u \mapsto 1 \otimes u$. More precisely, R-linearity means that $ru \mapsto g(r)(1 \otimes u) = g(r) \otimes u$ for all $r \in R$ and $u \in M$.

(11) Given homomorphisms $R \to S$ and $S \to T$, the base change functor $\mathcal{B}_{R\to T}$ for the composite homomorphism $R \to T$ is the composition $\mathcal{B}_{S\to T} \circ \mathcal{B}_{R\to S}$.

Part (1) is immediate from the definition. Part (2) holds because tensor product commutes with arbitrary direct sums and arbitrary direct limits. Part (3) is immediate from parts (1) and (2). If P is a projective R-module, one can choose Q such that $P \oplus Q$ is free. Then $(S \otimes_R P) \oplus (S \otimes_R Q)$ is free over S, and it follows that both direct summands are projective over S. Part (5) follows because if M is an R-module, the functor $(S \otimes_R M) \otimes_S \_$ on S-modules may be identified with the functor $M \otimes_R \_$ on S-modules. We have

$$(S \otimes_R M) \otimes_S U \cong (M \otimes_R S) \otimes_S U \cong M \otimes_R M,$$

by the associativity of tensor. Part (6) follows from the corresponding general fact for tensor products. Part (7) is immediate, for if M is finitely generated by n elements, we have a surjection $R^n \twoheadrightarrow M$, and this yields $S^n \twoheadrightarrow S \otimes_R M$. Part (8) is immediate from part (6), and part (9) is a consequence of (6) as well. (10) is completely straightforward, and (11) follows at once from the associativity of tensor products.

The Frobenius functors. Let R be a ring of positive prime characteristic p. The *Frobenius* or *Peskine-Szpiro* functor $\mathcal{F}_R$ from R-modules to R-modules is simply the base change functor for $f : R \to S$ when $S = R$ and the homomorphism $f : R \to S$ is the Frobenius endomorphism $F : R \to R$, i.e, $F(r) = r^p$ for all $r \in R$. We may take the e-fold iterated composition of this functor with itself, which we denote $\mathcal{F}_R^e$. This is the same as the base change functor for the homomorphism $F^e : R \to R$, where $F^e(r) = r^{p^e}$ for all $r \in R$, by the iterated application of (11) above. When the ring is clear from context, the subscript $_R$ is omitted, and we simply write $\mathcal{F}$ or $\mathcal{F}^e$.

We then have, from the corresponding facts above:

(1) $\mathcal{F}^e(R) = R$.

(2) $\mathcal{F}^e$ commutes with arbitrary direct sums and with arbitrary direct limits.

(3) $\mathcal{F}^e(R^n) = R^n$ and $\mathcal{F}^e$ takes free modules to free modules.

(4) $\mathcal{F}^e$ takes projective R-modules to projective R-modules.

(5) $\mathcal{F}^e$ takes flat R-modules to flat R-modules.

(6) $\mathcal{F}^e$ is right exact: if

$$M' \to M \to M'' \to 0$$

is exact, then so is

$$\mathcal{F}^e(M') \to \mathcal{F}^e(M) \to \mathcal{F}^e(M'') \to 0.$$

(7) $\mathcal{F}^e$ takes finitely generated modules to finitely generated modules: the number of generators does not increase.

(8) $\mathcal{F}^e$ takes the cokernel of the matrix $\big(r_{ij}\big)$ to the cokernel of the matrix $\big(r_{ij}^{p^e}\big)$.

(9) $\mathcal{F}^e$ takes R/I to $R/I^{[q]}R$.

By part (10) in the list of properties of base change, for every R-module M there is a natural map $M \to \mathcal{F}^e(M)$. We shall use u^q to denote the image of u under this map, which agrees with usual the usual notation when $M = R$. R-linearity then takes the following form:

(10) For every R-module M the natural map $M \to \mathcal{F}^e(M)$ is such that for all $r \in R$ and all $u \in M$, $(ru)^q = r^q u^q$.

We also note the following: given a homomorphism $g : R \to S$ of rings of positive prime characteristic p, we always have that $g \circ F_R^e = F_S^e \circ g$. In fact, all this says is that $g(r^q) = g(r)^q$ for all $r \in R$. This yields a corresponding isomorphism of compositions of base change functors:

(11) Let $R \to S$ be a homomorphism of rings of positive prime characteristic p. Then for every R-module M, there is an identification $S \otimes_R \mathcal{F}_R^e(M) \cong \mathcal{F}_S^e(S \otimes_R M)$ that is natural in the R-module M.

When $N \subseteq M$ the map $\mathcal{F}^e(N) \to \mathcal{F}^e(M)$ need not be injective. We denote that image of this map by $N^{[q]}$ or, more precisely, by $N_M^{[q]}$. *However, one should keep in mind that $N^{[q]}$ is a submodule of $\mathcal{F}^e(M)$,* **not** *of M itself.* It is very easy to see that $N^{[q]}$ is the R-span of the elements of $\mathcal{F}^e(M)$ of the form u^q for $u \in N$. The module $N^{[q]}$ is also the R-span of the elements u_λ^q as u_λ runs through any set of generators for N.

A very important special case is when $M = R$ and $N = I$, an ideal of R. In this situation, $I_R^{[q]}$ is the same as $I^{[q]}$ as defined earlier. What happens here is atypical, because $F^e(R) = R$ for all e.

Tight closure for modules Let R be a ring of positive prime characteristic p and let $N \subseteq M$ be finitely generated R-modules. If $N \subseteq M$, we define the *tight closure* N^*_M of N in M to consist of all elements $u \in M$ such that for some $c \in R^\circ$,

$$cu^q \in N^{[q]}_M \subseteq \mathcal{F}^e(M)$$

for all $q \gg 0$. Evidently, this agrees with our definition of tight closure for an ideal I, which is the case where $M = R$ and $N = I$. If M is clear from context, the subscript $_M$ is omitted, and we write N^* for N^*_M. Notice that we have not assumed that M or N is finitely generated. The theory of tight closure in Artinian modules is of very great interest. Note that c may depend on M, N, and even u. However, c is *not* permitted to depend on q. Here are some properties of tight closure:

Proposition. *Let R be a ring of positive prime characteristic p , and let N, M, and Q be R-modules.*

(a) *N^*_M is an R-module.*

(b) *If $N \subseteq M \subseteq Q$ are R-modules, then $N^*_Q \subseteq M^*_Q$ and $N^*_M \subseteq N^*_Q$.*

(c) *If $N_\lambda \subseteq M_\lambda$ is any family of inclusions, and $N = \bigoplus_\lambda N_\lambda \subseteq \bigoplus_\lambda M_\lambda = M$, then $N^*_M = \bigoplus_\lambda (N^*_\lambda)_{M_\lambda}$.*

(d) *If R is a finite product of rings $R_1 \times \cdots \times R_n$, $N_i \subseteq M_i$ are R_i-modules, $1 \le i \le n$, M is the R-module $M_1 \times \cdots \times M_n$, and $N \subseteq M$ is $N_1 \times \cdots \times N_n$, then N^*_M may be identify with $(N_1)^*_{M_1} \times \cdots \times (N_n)^*_{M_n}$.*

(e) *If I is an ideal of R, $I^* N^*_M \subseteq (IN)^*_M$.*

(f) *If $N \subseteq M$ and $V \subseteq W$ are R-modules and $h : M \to W$ is an R-linear map such that $h(N) \subseteq V$, then $h(N^*_M) \subseteq V^*_W$.*

Proof. (a) Let $c, c' \in R^\circ$. If $cu^q \in N^{[q]}$ for $q \ge q_0$, then $c(ru)^q \in N^{[q]}$ for $q \ge q_0$. If $c'v^q \in N^q$ for $q \ge q_1$ then $(cc')(u+v)^q \in N^{[q]}$ for $q \ge \max\{q_0, q_1\}$.

(b) The first statment holds because we have that $N^{[q]}_Q \subseteq M^{[q]}_Q$ for all q, and the second because the map $F^e(M) \to F^e(Q)$ carries $N^{[q]}_M$ into $N^{[q]}_Q$.

(c) is a straightforward application of the fact that tensor product commutes with direct sum and the definition of tight closure. Keep in mind that every element of the direct sum has nonzero components from only finitely many of the modules.

(d) is clear: note that $(R_1 \times \cdots \times R_n)^\circ = R_1^\circ \times \cdots \times R_n^\circ$.

(e) If $c, c' \in R^\circ$, $cf^q \in I^{[q]}$ for $q \gg 0$, and $c'u^{[q]} \in N^{[q]}$ for $q \gg 0$, then $(cc')(fu)^q = (cf^q)(c'u^q) \in I^{[q]}N^{[q]}$ for $q \gg 0$, and $I^{[q]}N^{[q]} = (IN)^{[q]}$ for every q.

(f) This argument is left as an exercise. □

Let R and S be Noetherian rings of positive prime characteristic p. We will frequently be in the situation where we want to study the effect of base change on tight closure.

For this purpose, when $N \subseteq M$ are R-modules, it will be convenient to use the notation $\langle S \otimes_R N \rangle$ for the image of $S \otimes_R N$ in $S \otimes_R M$. Of course, one must know what the map $N \hookrightarrow M$ is, not just what N is, to be able to interpret this notation. Therefore, we may also use the more informative notation $\langle S \otimes_R N \rangle_M$ in cases where it is not clear what M is. Note that in the case where $M = R$ and $N = I \subseteq R$, $\langle S \otimes_R I \rangle = IS$, the expansion of I to S. More generally, if $N \subseteq G$, where G is free, we may write NS for $\langle S \otimes_R N \rangle_G \subseteq S \otimes G$, and refer to NS as the *expansion* of N, by analogy with the ideal case.

Proposition. *Let $R \to S$ be a homomorphism of Noetherian rings of positive prime characteristic p such that R° maps into S°. In particular, this hypothesis holds (1) if $R \subseteq S$ are domains, (2) if $R \to S$ is flat, or if (3) $S = R/P$ where P is a minimal prime of S. Then for all modules $N \subseteq M$, $\langle S \otimes_R N^*_M \rangle_M \subseteq (\langle S \otimes_R N \rangle_M)^*_{S \otimes_R M}$.*

Proof. It suffices to show that if $u \in N^*$ then $1 \otimes u \in \langle S \otimes_R N \rangle^*$. Since the image of c is in S°, this follows because $c(1 \otimes u^q) = 1 \otimes cu^q \in \langle S \otimes_R N^{[q]} \rangle = \langle S \otimes_R N \rangle^{[q]}$.

The statement about when the hypothesis holds is easily checked: the only case that is not immediate from the definition is when $R \to S$ is flat. This can be checked by proving that every minimal prime Q of S lies over a minimal prime P of R. But the induced map of localizations $R_P \to S_Q$ is faithfully flat, and so injective, and QS_Q is nilpotent, which shows that PR_P is nilpotent. □

Tight closure, like integral closure, can be checked modulo every minimal prime of R.

Theorem. *Let R be a ring of positive prime characteristic p . Let $P_1, \ldots, P_n$ be the minimal primes of R. Let $D_i = R/P_i$. Let $N \subseteq M$ be R-modules, and let $u \in M$. Let $M_i = D_i \otimes_R M = M/P_iM$, and let $N_i = \langle D_i \otimes_R N \rangle$. Let u_i be the image of u in M_i. Then $u \in N^*_M$ over R if and only if for all i, $1 \le i \le n$, $u_i \in (N_i)^*_{M_i}$ over D_i.*

If $M = R$ and $N = I$, we have that $u \in I^*$ if and only if the image of u in D_i is in $(ID_i)^*$ in D_i, working over D_i, for all i, $1 \le i \le n$.

Proof. The final statement is just a special case of the Theorem. The "only if" part follows from the preceding Proposition. It remains to prove that if u is in the tight closure modulo every P_i, then it is in the tight closure. This means that for every i there exists $c_i \in R - P_i$ such that for all $q \gg 0$, $c_i u^q \in N^{[q]} + P_i F^e(M)$, since $\mathcal{F}^e(M/P_iM)$ working over D_i may be identified with $\mathcal{F}^e(M)/P_i\mathcal{F}^e(M)$. Choose d_i so that it is in all the P_j except P_i. Let J be the intersection of the P_i, which is the ideal of all nilpotents. Then for all i and all $q \gg 0$,

$$(*_i) \quad d_i c_i u^q \in N^{[q]} + JF^e(M),$$

since every $d_i P_i \subseteq J$.

Then $c = \sum_{i=1}^n d_i c_i$ cannot be contained in the union of P_i, since for all i the ith term in the sum is contained in all of the P_j except P_i. Adding the equations $(*_i)$ yields

$$cu^q \in N^{[q]} + JF^e(M)$$

for all $q \gg 0$, say for all $q \geq q_0$. Choose q_1 such that $J^{[q_1]} = 0$. Then $c^{q_1}u^{qq_1} \in N^{[qq_1]}$ for all $q \geq q_0$, which implies that $c^q u^q \in N^{[q]}$ for all $q \geq q_1 q_0$. $\square$

Let R have minimal primes $P_1, \ldots, P_n$, and let $J = P_! \cap \cdots \cap P_n$, the ideal of nilpotent elements of R, so that $R_{\mathrm{red}} = R/J$. The minimal primes of R/J are the ideals P_i/J, and for every i, $R_{\mathrm{red}}/(P_i/J) \cong R/P_i$. Hence:

Corollary. *Let R be a ring of positive prime characteristic p , and let J be the ideal of all nilpotent elements of R. Let $N \subseteq M$ be R-modules, and let $u \in M$. Then $u \in N^*_M$ if and only if the image of u in M/JM is in $\langle N/J \rangle^*_{M/JM}$ working over $R_{\mathrm{red}} = R/J$.*

We should point out that it is easy to prove the result of the Corollary directly without using the preceding Theorem.

We also note the following easy fact:

Proposition. *Let R be a ring of positive prime characteristic p . Let $N \subseteq M$ be R-modules. If $u \in N^*_M$, then for all $q_0 = p^{e_0}$, $u^{q_0} \in (N^{[q_0]})^*_{\mathcal{F}^{e_0}(M)}$.*

Proof. This is immediate from the fact that $(N^{[q_0]})^{[q]} \subseteq \mathcal{F}^e\big(\mathcal{F}^{e_0}(M)\big)$, if we identify the latter with $\mathcal{F}^{e_0+e}(M)$, is the same as $N^{[q_0q]}$. $\square$

We next want to consider what happens when we iterate the tight closure operation. When M is finitely generated, and quite a bit more generally, we do not get anything new. Later we shall develop a theory of *test elements* for tight closure that will enable us to prove corresponding results for a large class of rings without any finiteness conditions on the modules.

Theorem. *Let R be a ring of positive prime characteristic p , and let $N \subseteq M$ be R-modules. Consider the condtion :*

(#) *there exist an element $c \in R^\circ$ and $q_0 = p^{e_0}$ such that for all $u \in N^*$, $cu^q \in N^{[q]}$ for all $q \geq q_0$,*

which holds whenever N^*/N is a finitely generated R-module. If (#) holds, then $(N^*_M)^*_M = N^*_M$.

Proof. We first check that (#) holds when N^*/N is finitely generated. Let $u_1, \ldots, u_n$ be elements of N^* whose images generate N^*/N. Then for every i we can choose $c_i \in R^\circ$ and q_i such that for all $q \geq q_i$, we have that $c_i u^q \in N^{[q]}$ for all $q \geq q_i$. Let $c = c_1 \cdots c_n$ and let $q_0 = \max\{q_1, \ldots, q_n\}$. Then for all $q \geq q_0$, $cu_i^q \in N^{[q]}$, and if $u \in N$, the same condition obviously holds. Since every element of N^* has the form $r_1u_1 + \cdots + r_nu_n + u$ where the $r_i \in R$ and $u \in N$, it follows that (#) holds.

Now assume # and let $v \in (N^*)^*$. Then there exists $d \in R^\circ$ and q' such that for all $q \geq q'$, $dv^q \in (N^*)^{[q]}$, and so dv^q is in the span of elements w^q for $w \in N^*$. If $q \geq q_0$, we know that every $cw^q \in N^{[q]}$. Hence, for all $q \geq \max\{q',\, q_0\}$, we have that $(cd)v^q \in N^{[q]}$, and it follows that $v \in N^*$. $\square$

Of course, if M is Noetherian, then so is N^*, and condition (#) holds. Thus:

Corollary. *Let R be a ring of positive prime characteristic p , and let $N \subseteq M$ be finitely generated R-modules. Then $(N^*_M)^*_M = N^*_M$.* $\square$

The following result, used earlier without proof, is very useful in thinking about tight closure.

Proposition. *Let R be a ring of positive prime characteristic p , let $N \subseteq M$ be R-modules, and let $u \in M$. Then $u \in N^*_M$ if and only if the image $\overline{u}$ of u in the quotient M/N is in $0^*_{M/N}$.*

Hence, if we map a free module G onto M, say $h : G \twoheadrightarrow M$, let $H = h^{-1}(N) \subseteq G$, and let $v \in G$ be such that $h(v) = u$, then $u \in N^*_M$ if and only if $v \in H^*_G$.

Proof. For the first part, let $c \in R^0$. Note that, by the right exactness of tensor products, $\mathcal{F}^e(M/N) \cong \mathcal{F}^e(M)/N^{[q]}$. Consequently, $cu^q \in N^{[q]}$ for all $q \geq q_0$ if and only if $c\overline{u}^q = 0$ in $\mathcal{F}^e(M/N)$ for $q \geq q_0$.

For the second part, simply note that the image of v in $G/H \cong M/N$ corresponds to $\overline{u}$ in M/N. $\square$

It follows many questions about tight closure can be formulated in terms of the behavior of tight closures of submodules of free modules. Of course, when M is finitely generated, the free module G can be taken to be finitely generated with the same number of generators.

Given a free module G of rank n, we can choose an ordered free basis for G. This is equivalent to choosng an isomorphism $G \cong R^n = R \oplus \cdots \oplus R$. In the case of R^n, one may understand the action of Frobenius in a very down-to-earth way. We may identify $\mathcal{F}^e(R^n) \cong R^n$, since we have this identification when $n = 1$. Keep in mind, however, that the identification of $\mathcal{F}^e(G)$ with G depends on the choice of an ordered free basis for G. If $u = r_1 \oplus \cdots \oplus r_n \in R^n$, then $u^q = r_1^q \oplus \cdots \oplus r_n^q$. With $H \in R^n$, $H^{[q]}$ is the R-span of the elements u^q for $u \in H$ (or for u running through generators of H). Very similar remarks apply to the case of an infinitely generated free module G with a specified basis b_λ. The elements b_λ^q give a free basis for $\mathcal{F}^e(G)$, and if $u = r_1 b_{\lambda_1} + \cdots + r_s b_{\lambda_s}$, then $u^q = r_1^q b_{\lambda_1}^q + \cdots + r_s^q b_{\lambda_s}^q$ gives the representation of u^q as a linear combination of elements of the free basis $\{b_\lambda^q\}_\lambda$.

We earlier defined tight closure for submodules of free modules using this very concrete description of u^q and $H^{[q]}$. The similarity to the case of ideals in the ring is visibly very great. But we then have the problem of proving that the notion is independent of the choice of free basis. Moreover, with the earlier approach, we needed to define N^*_M by mapping a free module G onto M and replacing N by its inverse image in G. We then have the problem of proving that the notion we get is independent of the choices we make.