Math 615, Winter 2020: Lecture of Friday, April 17

We first discuss the tight closure proof that direct summands of regular rings are Cohen-Macaulay. This is the only part of this material that is not optional.

We then introduce the notion of integral dependence on an ideal and integral closure of ideals. A Supplement on integral closure has been provided. We then give the amazingly easy tight closure proof of a strengthend form of the Briançon-Skoda theorem. All of this material is optional in the sense that it will not be covered on the last quiz.

Direct summands of regular rings are Cohen-Macaulay

The following result application of tight closure theory is immediate from what we have already done.

Theorem. *A direct summand R of a weakly F-regular domain S is weakly F-regular. Hence, a direct summand of a regular ring of positive prime characteristic p that is a homomorphic image of a Cohen-Macaulay ring is Cohen-Macaulay.*

Proof. The first statement is part of the Theorem stated at the bottom of the first page of the Lecture of April 3. The second statement is the immediate from the final Theorem of the Lecture of April 8. $\square$

Again, material in the lecture notes beyond this point will not be tested in the remaining quiz.

Discussion. The restriction that R be a homomorphic image of a Cohen-Macaulay ring is not needed. One can localize R at a maximal ideal and then S at prime lying over the maximal ideal of R while maintaining the property that every ideal of R is contracted from S, and so assume that both are local. One can then complete R with respect to its maximal ideal $\mathfrak{m}$ and S with respect to $\mathfrak{m}S$ and so assume that R is complete, and that every ideal is contracted from S. Now R is a homomorphic image of a regular ring, and one has colon-capturing. If $x_1, \ldots, x_d$ is a system of parameters for R, and $I_i = (xi_1, \ldots, i_)$ for $0 \leq i \leq d-1$, then $I_i : x_{i+1} \subseteq I*$ in R, and this is contained in $(I_iS)^*$ in S and in R. But $(I_iS)^* = IS$ since S is regular, and $I_iS \cap R = I_i$. $\square$

The extension of tight closure theory to rings containing the rational numbers gives a proof of the same result for rings containing the rationals, and the result has recently been extended to mixed characteristic by perfectoid methods.

The Briançon-Skoda theorem

This section is optional.

We first discuss the notion of the integral closure of an ideal in the Noetherian case.

Theorem. *Let R be Noetherian, I an ideal, and let $r \in R$. The following conditions are equivalent, and define the condition that an element $r \in R$ be integral over an ideal I of R.*

(1) *There is an element c not in any minimal prime of R such that $cr^n \in I^n$ for all $n \gg 0$ (equivalently, for infinitely many values of n).*
(2) *There is an element c not in any minimal prime of R such that $cr^n \in I^n$ for infinitely many values of n.*
(3) *For every map of R into a Noetherian discrete valuation domain V, the image of r is IV.*
(4) *For every minimal prime $\mathfrak{p}$ of R and Noetherian discrete valuation domain between $R/\mathfrak{p}$ and its fraction field, the image of r is in IV.*
(5) *rt is integral over the Rees ring $R[It] \subseteq R[t]$ (the latter is the polynomial ring in one variable t over$R$).*
(6) *For some positive integer n, the element r satisfies a polynomial equation of the form*
$$r^n + i_1 r^{n-1} + \cdots + i_j r^{n-j} + \cdots + i_{n-1} r + i_n = 0$$
where the coefficient $i_j \in I^j$.

Note that it suffices if $r^n \in I^n$: let $i_n := r^n$, and use the equation $r^n - i_n = 0$.

For our purposes here, we will take (2) as the definition of when an element is integral over an ideal in the Noetherian case. When R is not Noetherian, (5) and (6) are equivalent and imply the other conditions and either may be taken as the definition of when r is in the integral closure of I. In all cases, the set of elements integral over I is an ideal called the *integral closure* of I, and denoted $\overline{I}$. There is a treatment of integral closure of ideals in a new Supplement on the Web page. There is a great deal more on the subject in the Lecture Notes for Math 615, Winter 2019.

We first note:

Theorem. *Let R be a Noetherian ring of positive prime characteristic p. If $r \in I^*$, then $r \in \overline{I}$. In other words, $I^* \subseteq \overline{I}$.*

Tight closure is typically much smaller than integral closure. For example in $K[x, y]$ or $K[[x, y]]$, where x, y are indeterminates, the ideal (x^n, y^n) is tightly closed for all integers n. But its integral closure contains $(x, y)^n$, since if $i+j = n$, $(x^i y^j)^n = (x^n)^i (y^n)^j \in (x^n, y^n)^n$.

The following result was first proven for algebras over the complex numbers and convergent power series rings over the complex numbers by analytic methods.

Theorem (Briançon-Skoda). *If I is an ideal of a regular ring and is generated by n elements, then $\overline{I^n} \subseteq I$.*

There are many refined versions, but, for simplicity, we only consider this statement here.

Later, an algebraic proof of the Briançon-Skoda theorem was given by J. Lipman and A. Sathaye that is valid in all characteristics, including mixed characteristic: we refer to the Lecture Notes from Math 615, Winter 2019 for a full treatment.

Tight closure theory permits an extremely simple proof of a stronger result in the case where the ring contains a field, which we want to give here. We first want to note two corollaries of the Briançon-Skoda theorem, but we refer to the Lecture Notes from Math 615, Winter 2019 for the details of how they follow from it.

Corollary. *Suppose that $f \in \mathbb{C}\{z_1, \ldots, z_n\}$ is a convergent power series in n variables with complex coefficients that defines a hypersurface with an isolated singularity at the origin, i.e., f and its partial derivatives $\partial f/\partial z_i$, $1 \le i \le n$, have an isolated common zero at the origin. Then f^n is in the ideal generated by the partial derivatives of f in the ring $\mathbb{C}\{z_1, \ldots, z_n\}$.*

This answers affirmatively a question raised by John Mather.

Second:

Corollary. *Let $f_1, \ldots, f_{n+1}$ be polynomials in n variables over a field. Then $f_1^n \cdots f_n^n \in (f_1^{n+1}, \ldots, f_{n+1}^{n+1})$.*

For example, when $n = 2$ this implies that if $f, g, h \in K[x, y]$ are polynomials in two variables over a field K then $f^2g^2h^2 \in (f^3, g^3, h^3)$. This statement is rather elementary: the reader is challenged to prove it by elementary means.

Here is the tight closure version in characteristic $p > 0$.

Theorem (Generalized Briançon-Skoda theorem). *Let R be a ring of positive prime characteristic p . Let $I = (f_1, \ldots, f_n)$ be an ideal of R generated by n elements. Then $\overline{I^n} \subseteq I^*$.*

Proof. Suppose $r \in \overline{I^n}$ and c is an element not in any minimal prime of R such that $cr^h \in I^h$ for all $h \gg 0$. Then when $h = q = p^e \gg 0$ we have $cr^q \in \big((f_1, \ldots, f_n)^h\big)^q = (f_1, \ldots, f_n)^{nq} \subseteq (f_1^q, \ldots, f_n^q)$ because, in a monomial of degree nq in n elements, at least one of the exponents on one of the elements must be at least q. Hence, $cr^q \in I^{[q]}$ for all $q \in 0$. $\square$

We now recover the usual Briançon-Skoda theorem not just for regular rings, but for every weakly F-regular ring, since in that case $I^* = I$.

Remark. One easily gets the same result for algebras containing the rational numbers using the notion of tight closure in equal characteristic 0 that was discussed briefly earlier, and this recovers the original Briançon-Skoda theorem.