Math 615, Winter 2020: Lecture of Monday, April 20

Let $1 \leq t \leq r \leq s$ and let K be a field. This final lecture is devoted to proving that over any field K, if one considers the polynomial ring $S = K[x_{ij} : 1 \leq i \leq r,\, 1 \leq j \leq s]$ in the entries of an $r \times s$ matrix $X = \bigl(x_{ij}\bigr)$ then $I_t(X)$, the ideal generated by the $t \times t$ minors of X is prime, and the ring $S/I_t(X)$ is a Cohen-Macaulay domain. It is, in fact, also normal.

There are several ways to approach the problem of proving that large classes of ideals are prime. One is the method of Hodge algebras (also called algebras with straighten law), and you can read about them in the book of W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics **39**, Revised Edition, Cambridge University Press, 1993. We shall prove the result here using a different method: that of *principal radical systems*, initially developed in J. A. Eagon and M. Hochster, *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math. **93** (1971) 1020–1058. This method typically involves enlarging the class of ideals considered to a very large family consisting of radical ideals, containing the prime ideals of the primary decomposition of every ideal in the family, and with the property that whenever I is in the family, there is an ideal in the family generated by I and one additional element. One then establishes that the family consists entirely of radical ideals by what amounts to Noetherian induction: one proves the result for a given ideal in the family assuming it for all the larger ideals in the family. The base of the induction is easy because the maximal elements of the family are generated by subsets of the indeterminates. One uses the fact that $I + fS$ is radical to deduce that I is radical. This leads to an understanding of all of the ideals and of their primary decompositions. The application is usually to finitely generated graded algebras over a field K.

The Cohen-Macaulay property for the primes and some of the radical ideals in the family is then deduced, again by Noetherian induction, by one of two methods. In some cases, one uses that there is a nonzerodivisor f modulo I such that $I + fS$ is in the family. One can then deduce the Cohen-Macaulay property for S/I from the Cohen-Macaulay property for $S/(I + fR) \cong (S/I)/f(S/I)$, where one knows inductively that for the larger ideal $I + fS$, the quotient $S/(I + fS)$ is Cohen-Macaulay. In other cases, one deduces that $S/(I \cap J)$ is Cohen-Macaulay from the Cohen-Macaulay property for the quotients by the larger ideals I, J, and $I + J$. There are also techniques that work for more complicated intersections, but we will not need them for the case of determinantal ideals. In the situation where one has homogenous ideals I, J in a polynomial ring. it turns out that if I and J both have height h while $I + J$ has height $h + 1$ and all of S/I, S/J, and $S/(I + J)$ are Cohen-Macaulay, then $S/(I \cap J)$ is also Cohen-Macaulay. This is deduced using the fact that there is a short exact sequence

$$0 \to S/(I \cap J) \to (S/I) \oplus (S/J) \to S/(I + J) \to 0$$

along with standard facts about depth: see page 213 (the fifth page of the notes for this lecture), and its Corollary on the following page.

For pedagogical reasons we first give the proof that ideals of minors define Cohen-Macaulay domains for the case of 2×2 minors, and then we consider the general case. This is not efficient, but should make the ideas of the argument clearer.

The method of principal radical systems is based on two simple lemmas, stated below.

Lemma. *Let R be a Noetherian ring that is either local or $\mathbb{N}$-graded, and let $x \in R$ be in the maximal ideal or be a form of positive degree. Suppose that N is the nilradical of R, that N is prime, that $x \notin N$, and that R/xR is reduced. Then $N = 0$, i.e., R is a domain.*

Proof. Suppose that $u \in N$. Since R/xR is reduced, we must have that $u = xv$ for some $v \in R$. Since $xv \in N$, $x \notin N$, and N is prime, we must have that $v \in N$. Therefore $N = xN$. By Nakayama's lemma for local or graded rings, $N = 0$. $\square$

By applying this Lemma to R/I in the situation below, we obtain:

Corollary. *Let R be a Noetherian ring that is either local or $\mathbb{N}$-graded, and let $x \in R$ be in the maximal ideal or be a form of positive degree. Suppose that I is a (homogeneous in the graded case) proper ideal of R with radical P, where P is prime, that $x \notin P$, and that $P + xR$ is radical. Then $I = P$, i.e., I is prime.*

The next Lemma has various generalizations that may prove useful, but we shall stick with the simplest case.

Lemma. *Let R be Noetherian, let I be an ideal of R, let J be the radical of I, and suppose that $J \subseteq P$ where P is prime. Suppose that $I+xR$ is radical where $x \notin P$, and that $xP \subseteq I$. Then $I = J$, i.e., I is radical.*

Proof. Suppose that $u \in J$. Then $u \in I + xR$, say $u = i + xr$, where $i \in I$ and $r \in R$. Then $xr = u - i \in J \subseteq P$, and so $r \in P$. Since $xP \subseteq I$, we have that $xr \in I$ and so $u = i + xr \subseteq I$. $\square$

We want to use these lemmas to prove the following result:

Theorem. *Let K be a field, let r and s be positive integers, let t be an integer with $1 \leq t \leq \min\{r, s\}$, and let X be an $r \times s$ matrix of indeterminates over K. Then $I_t(X)$ is a prime ideal, i.e., $K[X]/I_t(X)$ is a domain.*

The proof will take a while. The idea is to include $I_t(X)$ in a much larger, but finite, family of ideals to which we can apply the lemmas above. The ideals are typically radical rather than prime. The result is proved by reverse induction, in that the largest ideal(s) in the family are shown to be radical first. The family has the property that for each ideal I in it that is not maximal in the family, there is a larger ideal of the form $I + xR$ in the family, which will be known to be radical from the induction hypothesis.

We shall show first that the ideals $I_t(X)$ have radicals that are prime. Thus, once we show that they are radical, it will follow that they are prime.

Note that if L is the algebraic closure of K and R is a K-algebra, $R \subseteq L \otimes_K R$, and so to show that R is reduced or a domain it suffices to show the corresponding fact for $L \otimes_K R$. Thus, the problem we are discussing reduces to the case where K is algebraically closed, and we assume this from here on. This will enable us to take a naive approach to the material we need from algebraic geometry, which will involve only basic facts about closed algebraic sets in affine spaces $\mathbb{A}^N_K$.

Note that showing that $\mathrm{Rad}\,(I_t(X))$ is prime is equivalent to showing that $V(I_t(X))$ is an irreducible closed algebraic set. We think of points of $\mathbb{A}^{rs}_K$ as corresponding to $r \times s$ matrices over K. Then $V(I_t(X))$ is precisely the set of $r \times s$ matrices of rank $\leq t-1$.

Proposition. *Let r, s, and t be as above. Let A be an $r \times s$ matrix over a field K. Then A has rank $\leq t-1$ if and only if A factors BC where B is an $r \times (t-1)$ matrix over K and C is a $(t-1) \times s$ matrix over K.*

Proof. We think of A as giving a linear map $K^s \to K^r$, where K^s is interpreted as $s \times 1$ columns. The rank is at most $t-1$ if and only if the image has dimension $\leq t-1$, i.e., if and only if the map factors $K^s \to K^h \to K^r$ where $h \leq t-1$. We may think of K^{t-1} as $K^h \oplus K^{t-1-h}$ and extend the map $K^h \to K^r$ to the additional summand K^{t-1-h} by letting it be 0. This gives a factorization $K^s \to K^{t-1} \to K^r$ for A which yields that $A = BC$, as required, while any linear map with such a factorization obviously has rank at most $t-1$. □

Corollary. *With notation as above, $V(I_t(X))$ is irreducible.*

Proof. Think of $\mathbb{A}^{(r+s)(t-1)} \cong \mathbb{A}^{r(t-1)}_K \times \mathbb{A}^{(t-1)s}_K$ as indexing pairs of matrices $(B,\,C)$ where B is $r \times (t-1)$ and C is $(t-1) \times s$. We have a map $\mathbb{A}^{(r+s)(t-1)} \to V((I_t(X))$ that sends $(B,\,C) \mapsto BC$, and by the preceding Proposition this map is surjective. Since $\mathbb{A}^{(r+s)(t-1)}$ is irreducible and the image of an irreducible is irreducible, $V(I_t(X))$ is irreducible. □

Of course, this establishes that $\mathrm{Rad}\,(I_t(X))$ is prime.

For heuristic reasons, we now carry through the proof that $I_t(X)$ is radical first for the case where $t = 2$. Let $J_{k,h,a}(X) = J_{k,h,a}$ denote the ideal generated by the entries of the first h rows of X, the first k columns of X, and the first a entries of the $(h+1)$ st row of X. Here, $0 \leq k \leq s$, $0 \leq h \leq r$, and $0 \leq a \leq s$. If $h = r$ or $k = s$ all the variables have been killed and $a = 0$ is forced. We also abbreviate $J_{k,h,0} = J_{k,h}$ and $J_{0,0,a} = J_a$. Note that $J_{k,h,a} = J_{k,0} + J_{0,h,a}$. If $a \leq k$, $J_{k,h,a} = J_{k,h}$. Certain ideals have more than one description: e.g., if $h < r$, $J_{k,h,s} = J_{k,h+1,0}$.

We shall prove by induction that all of the ideals $I_2(X) + J_{k,h,a}(X)$ are radical, and prime if $a = 0$. We assume the result for smaller matrices of indeterminates. Evidently, $I_2(X) + J_{s,r} = J_{s,r}$ is the ideal generated by all the indeterminates and is maximal. We now consider one ideal $I = I_2(X) + J_{k,h,a}(X)$ in the family, and assume that all larger ideals in the family are radical. We need to show that I is radical.

We can simplify things a bit as follows. Let X' be the $(r-h) \times (s-k)$ matrix in the lower right corner of X. As noted above we may assume that $a \geq k$. Then we have an obvious isomorphism

$$K[X]/\big(I_2(X) + J_{k,h,a}(X)\big) \cong K[X']/\big(I_2(X') + J_{a-k}(X')\big)$$

induced by the K-algebra surjection $K[X] \twoheadrightarrow K[X']$ that fixes each indeterminate in X' while sending the other indeterminates to 0. Since we know the result for the smaller matrix X' if either h or k is positive, there is no loss of generality in assuming that $h = k = 0$. Likewise, we may assume that $0 \leq a \leq s-1$. Finally, if either r or s is 1, then $I_2(X) = 0$, and the ideal is generated by a subset of the variables and is clearly prime. Henceforth we assume that $r, s \geq 2$.

Thus, $I = I_2(X) + J_a$ where $0 \leq a \leq s-1$. Let $x = x_{1,a+1}$. Since we know that larger ideals in the family are radical, we have that $I_2(X) + J_{a+1}$ is radical, and this is $I + (x)$. We consider two cases.

(1) $a = 0$. In this case, we know that $\mathrm{Rad}\,(I)$ is prime. The result now follows from the corollary to the first lemma, provided that we know that x is not in the radical of I. This follows because we can specialize x_{11} to 1 and all other variables to 0 and we get a point of $V(I)$ where $x_{11} \neq 0$. $\square$

(2) $1 \leq a \leq s-1$. For every i,j such that $2 \leq i \leq r$, $1 \leq j \leq a$, consider the 2×2 submatrix of X formed by the intersection of the first and $i\,th$ rows of X with the j th and $a+1$ st columns, namely:

$$\begin{pmatrix} x_{1,j} & x_{1,a+1} \\ x_{i,j} & x_{i,a+1} \end{pmatrix}.$$

The determinant of this matrix is in $I_2(X)$, and so $x_{1,j}x_{i,a+1} - x_{i,j}x \in I_2(X) \subseteq I$. Since $x_{1,j} \in J_a \subseteq I$ as well, we have that $xx_{i,j} \in I$. Let $P = I_2(X) + J_{a,0}(X)$. This is a larger ideal of our family, and is therefore radical, by the induction hypothesis. But the quotient by it is $\cong K[X']/I_2(X')$, where X' is the submatrix of X formed by the last $s-a$ columns of X, and so the radical is prime. Thus, P is a prime ideal containing J, and is generated over I by the elements $x_{i,j}$, $2 \leq i \leq r$, $1 \leq j \leq a$. It follows that $xP \subseteq I$. Finally, $x \notin P$, since we get a point of $V(P)$ by specializing so that $x_{1,a+1} = 1$ while every other indeterminate is specialized to 0. The fact that I is radical now follows from the second lemma. $\square$

We next want to use the work that we have done on the ideals $I_2(X)$ to prove that the rings $K[X]/I_2(X)$ are all Cohen-Macaulay rings. We first need to calculate the dimensions of the rings $K[X]/I_2(X)$.

Proposition. *Let X be an $r \times s$ matrix of indeterminates. The localization of $R = K[X]/I_2(X)$ at the element $x = x_{1,1}$ is isomorphic with the localization of the polynomial ring $S = K[x_{i,1}, x_{1,j} : 1 \leq i \leq r,\ 1 \leq j \leq s]$ at the element $x = x_{1,1}$. Hence,* $\dim(R) = \dim(R_x) = r + s - 1$.

Proof. For $i \geq 2$, $j \geq 2$ the equation given by the vanishing of the 2×2 minor formed the first and i th rows and the first and j th columns is

$$x_{1,1}x_{i,j} - x_{1,j}x_{i,1} = 0$$

which is equivalent to

$$(*) \quad x_{i,j} = x_{1,j}x_{i,1}/x$$

in R_x. Consider the K-algebra homomorphism $K[X] \to S_x$ that fixes $x_{i,j}$ if $i = 1$ or if $j = 1$, and otherwise sends $x_{i,j} \mapsto x_{1,j}x_{i,1}/x$. It is straightforward to verify that the map kills $I_2(X)$ and so induces a surjection $R_x \twoheadrightarrow S_x$. The inclusion $S \subseteq K[X]$ induces a map $S_x \to R_x$. The composition $(R_x \to S_x) \circ (S_x \to R_x)$ is clearly the identity on S_x, and the composition $(S_x \to R_x) \circ (R_x \to S_x)$ is the identity on R_x because of the displayed relations $(*)$. The statement about dimensions is clear, since $\dim(S_x) = \dim(S) = r + s - 1$. □

We may also argue as follows in calculating the dimension of R, which is the same as the dimension of $V(I_2(X))$. Consider the open set where the first row of the matrix is nonzero. The first row varies in $\mathbb{A}^s_K$ (with the origin deleted), i.e., in a variety of dimension s. Each of the other $r - 1$ rows is a scalar times the first row, and so one expects the dimension to be $s + (r - 1) = r + s - 1$.

Theorem. *With r, s, X as above, each of the rings $K[X]/I_2(X)$ is Cohen-Macaulay.*

Before proving this, we note the following.

Lemma. *If I and J are any ideals of any ring R, there is an exact sequence:*

$$0 \to R/(I \cap J) \to R/I \oplus R/J \to R/(I + J) \to 0$$

where the first map sends $r + (I \cap J) \mapsto (r + I) \oplus (-r + J)$ and the second map sends $(r + I) \oplus (r' + J) \mapsto (r + r') + (I + J)$.

Proof. It is straightforward to check that the maps are well-defined and R-linear. The first map is injective, since $r + (I \cap J)$ is in the kernel iff $r \in I$ and $r \in J$. The second map obviously kills the kernel, and is clearly surjective, Finally, $(r+I) \oplus (r'+J)$ is in the kernel of the second map iff $r + r' \in I + J$, i.e., $r + r' = i + j$ with $i \in I$ and $j \in J$. But then $r - i = r' + j = r_0$, and $(r + I) \oplus (r' + J)$ is the image of $r_0 + (I \cap J)$. □

In the case of an $\mathbb{N}$-graded Noetherian ring R with $R_0 = K$, a field, and homogeneous maximal ideal m, R is Cohen-Macaulay if and only if $\text{depth}_m R = \dim(R)$. We also have:

Corollary. *Let hypotheses be as in the preceding Lemma. Suppose that R is a finitely generated $\mathbb{N}$-graded algebra over a field K with $R_0 = K$ and that I, J are ideals such that R/I, R/J are Cohen-Macaulay of dimension $d - 1$ and $R/(IJ)$ is Cohen-Macaulay of dimension $d - 2$. Then $R/(I \cap J)$ is Cohen-Macaulay of dimension $d - 1$.*

Proof. We need only check that the depth of $R/(I \cap J)$ i on its maximal ideal, or on $\mathfrak{m}$, the maximal ideal of R, is $d-1$. Since the depth $R/(I+J)$ on $\mathfrak{m}$ is $d-2$ and the depth of $(R/I) \oplus (R/J)$ on $\mathfrak{m}$ is $d-1$, this is immediate from the long exact sequence for Ext. $\square$

Proof of the Theorem. We use induction, and so we may assume the result if either or both of the dimensions of the matrix X are decreased. R is a domain and $x = x_{1,1}$ is therefore a nonzerodivisor. It will therefore suffice to prove that R/xR is Cohen-Macaulay. In R/xR, the fact that $x_{i,1}x_{1,j} - x_{1,1}x_{i,j} = 0$ shows that any minimal prime of x either contains all the $x_{1,j}$ or all of the $x_{i,1}$. Let P be the ideal $I_2(X) + (x_{1,j} : 1 \leq j \leq s)$ and Q the ideal $I_2(X) + (x_{i,1} : 1 \leq i \leq r)$. It follows that $V(x) = V(P) \cup V(Q)$. Since all of these ideals are radical, we have that $xR = P \cap Q$.

Let X', X'', and X''' be the matrices obtained from X be deleting, respectively, the first row, the first column, and both the first column and row. Then $R/P \cong K[X']/I_2(X')$ is a Cohen-Macaulay domain of dimension $(r-1)+s-1 = r+s-2$ by the induction hypothesis. $R/Q \cong K[X'']/Q$ is, similarly, a Cohen-Macaulay domain of dimension $r + (s-1) - 1 = r + s - 2$. Moreover, $K[X]/(P+Q) \cong K[X''']/I_2(X''')$ is a Cohen-Macaulay domain of dimension $(r-1) + (s-1) - 1 = r + s - 3$, again using the induction hypothesis. We can now make of the short exact sequence

$$0 \to R/xR \to R/P \oplus R/Q \to R/(P+Q) \to 0.$$

Since the module in the middle has depth $r + s - 2$ on m and the module on the right has depth $r + s - 3$ on m, the module on the left has has depth $r + s - 2$ on m. (One may use the long exact sequence for $\mathrm{Ext}_R(K,\ \_)$, or for Koszul homology, or for local cohomology to show this.) Since x is not a zerodivisor in the domain R, it follows that $\mathrm{depth}_m(R) = r + s - 1$, which is $\dim(R)$. Therefore, R is Cohen-Macaulay. $\square$

We now want to generalize all this to the case of $t \times t$ minors. We introduce two notations that will be useful in dealing with matrices. If A is a matrix, we write $A|_t$ for the submatrix formed from the first h columns of A. If A and B are matrices of sizes $r \times t$ and $r \times u$, respectively, we write $A \# B$ for the $r \times (t+u)$ matrix obtained by *concatenating* A and B: the first t columns of $A \# B$ give A, while the last u columns give B.

The following elementary fact will prove critical in our analysis. It generalizes the fact that when the two by two minors of a matrix vanish and the entries of the first row in the first v columns are 0, then the rest of the entries of the first row kill the elements in the first v columns.

Lemma (killing minors). *Let $A = (a_{ij})$ be a matrix and $1 < v < w$ integers such that the $(k+1) \times (k+1)$ minors of $A|_w$ vanish. Suppose also that $a_{1j} = 0$ for $1 \leq j \leq v$. Then for $v < j \leq w$, a_{1j} kills $I_k(A|_v)$.*

Proof. Fix j and fix a $k \times k$ minor of $A|_v$. If the minor involves the first row of A, it is 0, since the first row of $A|_v$ is 0. Therefore we may assume that the minor involves k rows of A other than the first and k columns of A that are actually columns of $A|_v$. Let

B denote the $k \times k$ submatrix of A determined by these rows and columns. Consider the $(k+1) \times (k+1)$ submatrix of A that involves, additionally, the first row of A and the j th column of A. This submatrix has the block form $\begin{pmatrix} 0 & a_{1,j} \\ B & C \end{pmatrix}$ where 0 denotes a $1 \times k$ block and C denotes a $k \times 1$ block. The determinant of this matrix is 0 by hypothesis, and is equal to $\pm a_{i,j} \det(B)$. The result follows. $\square$

Now suppose that we want to create a family of ideals that can be used to prove that ideals of the form $I_3(X)$ are prime. If we kill the variables in the first v columns of the first row, we are led to consider ideals in which the 2×2 minors of the first v columns are 0 and the 3×3 minors of the entire matrix are 0. In addition, some of the entries of the first row are 0. Eventually we may lose the entire first row.

When we consider building an appropriate family for $I_4(X)$, we are led to consider ideals of the form $I_3(X|_v) + I_4(X) + I_a(X)$. But once the first row is gone, and we start to kill entries of te second row, we see that we need to consider ideals of the form $I_2(X|_u) + I_3(X_v) + I_4(X) + I_a(X)$. This suggests studying the large class that we are about to introduce.

Let X be an $r \times s$ matrix and $1 \leq t \leq \min r,\, s$ as before. Let $\sigma = (s_0,\, s_1,\, \ldots,\, s_{t-1})$, where the s_j are nonnegative integers $\leq s$ and $s_{t-1} = s$. We denote by $I_\sigma(X)$ the ideal

$$I_1(X|_{s_0}) + I_2(X|_{s_1}) + \cdots + I_t(X|_{s_{t-1}}).$$

We shall prove:

Theorem. *Let K be a field, and X an $r \times s$ matrix of indeterminates over K with r, s, t as above. Then all of the ideals $I_\sigma(X) + I_{k,h,a}(X)$ are radical, where $\sigma = (s_1,\, \ldots,\, s_{t-1})$ as above.*

We shall also prove that certain ideals among these are prime, and the the quotients by these primes are Cohen-Macaulay, but before we state the precise result, we want to introduce some restrictions on the elements σ and k, h, a used to describe the ideals.

Exactly as in our analysis of the case where $t = 2$, if $k > 0$ or $h > 0$ we can consider instead an ideal of the same type defined using a matrix of indeterminates with at least one dimension strictly smaller than r or s. Henceforth, we assume that $h = k = 0$. Having a positive value for s_0 has the same effect as having the same value for k. We may likewise assume that $s_0 = 0$. If we are not killing any $j \times j$ minors in our sum $I_\sigma(X)$, we assume that $s_{j-1} = j - 1$. Note that $X|_{j-1}$ has rank at most $j - 1$ automatically. Also note that if the size j minors vanish for the first u columns, the same is true for the size $j+1$ minors. This enables us to assume that $s_{j-1} \leq s_j$. But we can say more: the size $j+1$ minors will vanish for $X|_{u+1}$ as well, since a $j+1$ size minor that involves the last column may be expanded with respect to that column, and the cofactors are size k minors of $X|_u$. Henceforth, we may assume without loss of generality that $0 < s_1 < s_2 \cdots < s_{t-1} = s$. When this condition holds, we shall say that σ is *standard*. Note that when we want to work with $I_t(X)$, we work instead with $I_\sigma(X)$ for $\sigma = (0, 1, 2, 3, \ldots, t-2, s)$.

We can now state a more precise version of the theorem that we are aiming to prove.

Theorem. *Let K, X, r, s, t, k, h, σ, and a be as above. Then $I_\sigma(X) + J_{k,h,a}(X)$ is radical.*

If σ is standard and $a = s_k$ for some k (the case where $k = 0$, when $s_k = 0$, is included), then $P = I_\sigma(X) + I_{s_k}(X)$ is prime, and the ring $K[X]/P$ is Cohen-Macaulay.

The proof will occupy as for a while, but is, in fact, quite similar to the argument for the case where $t = 2$.

We first prove:

Lemma. *Let $0 \le k < t \le r$ be integers and K a field. Let L be a nonzero linear functional on K^r and let $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{t-1}$ be a nondecreasing chain of subspaces of K^r such that L vanishes on V_k (hence, on all of $V_1, \ldots, V_k$) and $\dim_K(V_j) \le j$ for $1 \le j \le t-1$. Then there exists a chain of subspaces $0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_{t-1}$ in K^r such that L vanishes on W_k, $V_j \subseteq W_j$ for $1 \le j \le t-1$, and $\dim(W_j) = j$ for $1 \le j \le t-1$.*

Proof. We construct the W_j by reverse induction on j. We may evidently choose $W_{t-1} \subseteq K^r$ such that $V_{t-1} \subseteq W_{t-1}$ and $\dim_K(W_{t-1}) = t-1$, since $\dim(V_{t-1}) \le t-1 < r$. If $W_{j+1}, \ldots, W_{t-1}$ have already been chosen satisfying the required conditions, $j > 1$, then there are two cases. If $j \neq k$, simply chose W_j of dimension j lying between $V_j \subseteq W_{j+1}$, which is possible since $\dim(V_j) \le j$ and $\dim(W_{j+1}) = j+1$. If $j = k$, let H denote the kernel of L, a codimension one subspace of K^r. We now have to choose W_k of dimension k so that it contains V_k and is contained in $H \cap W_{k+1}$. But the dimension of $H \cap W_{k+1} \ge \dim(H) + \dim(W_{k+1}) - r = r - 1 + k + 1 - r = k$, and since $V_k \subseteq H \cap W_{k+1}$ and has dimension at most k, this is possible. $\square$

We recall some facts about affine algebraic varieties over an algebraically closed field K. If X is an affine algebraic set over K, its coordinate ring $K[X]$ is the same as the set of regular functions from X to K. If $I \subseteq K[x_1, \ldots, x_n]$, a polynomial ring, is the radical ideal that defines XS, so that $X = \mathcal{V}(I)$, then $K[X] \cong K[x_1, \ldots, x_n]/I$. X is a variety if and only if I is prime, and also if and only if $K[X]$ is an integral domain. There is an antiequivalence of categories between affine algebraic sets (which arise as Zariski closed subsets of $\mathring{A}^n_K$ and regular morphism reduced finitely generated K-algebras. The algebraic set X corresponds to the ring $K[X]$. A regular morphism of algebraic varieties $X \to Y$ is called *dominant* if the image of X is Zariski dense in Y. This is equivalent to saying that the corresponding map of coordinate rings $K[Y] \to K[X]$ is an injective homomorphism of domains.

In our discussion of dimension in the sequel we need a fact relating the dimension of the domain of a dominant morphism to the dimension of the image and the dimension of a "typical" fiber. We treat this result formally below, but we first need some important facts about flatness. Part (a) is a special case of the Theorem on Generic Freeness proved in a stronger form in the notes for the Lectures from March 16–18, p. 134. We give the proof of part (b).

Lemma. *Let A, R, and S be Noetherian rings.*

(a) *(Generic freeness) Let A be a domain. If M is a finitely generated module over a finitely generated A-algebra R, then there exists $a \in A - \{0\}$ such that M_a is free over A_a.* $\square$

(b) *If $(R,\, m,\, K) \to (S,\, \mathfrak{n},\, L)$ is a flat local homomorphism of local rings, then* $\dim(S) = \dim(R) + \dim(S/\mathfrak{m}S)$.

Proof. For part (b), let J be nilradical of R. Then $R/J \to S/JS$ is again a flat local homomorphism, the dimensions don't change, since we are killing ideals of nilpotents, S/mS is the same as $(S/JS)/(m/J)S/(JS)$, since $J \subseteq m$. Thus, we may assume that R is reduced. We use induction on $\dim(R)$. If $\dim(R) = 0$, then since R is reduced, we have $R = K$, and $S/mS \cong S$. The result is now obvious. If $\dim(R) = 1$, we can use prime avoidance to choose and element $x \in \mathfrak{m}$ not in any minimal prime of R, and since R is reduced, $x \notin \operatorname{Ass}(R)$. Hence, x is not a zerodivisor on R. Since S is R-flat, x is not a zerodivisor in S. It follows that $R/xR \to S/xS$ is flat, and $(S/xR)/(\mathfrak{m}/xR)(S/xS) \cong S/\mathfrak{m}S$. We have $\dim(R/xR) = \dim(R) - 1$, $\dim(S/xS) = \dim(S) - 1$, and the induction hypothesis yields that $\dim(S) - 1 = \dim(R-1) + \dim(S/\mathfrak{m}S)$, and the desired conclusion follows at once. $\square$

The following result gives some properties of dominant maps of algebraic varieties. Throughout, K is an algebraically closed field.

Lemma. *Let $g : X \to Y$ be a dominant map of algebraic varieties, so that we have an injection of domains $K[Y] \hookrightarrow K[X]$. Then:*

(a) *The transcendence degree of $K(X)$ over $K(Y)$ is $\delta = \dim(X) - \dim(Y)$.*

(b) *There is a dense open subset U of Y such that for every $u \in U$, the dimension of the fiber $g^{-1}(u)$, thought of as a closed algebraic set in X, is $\delta = \dim(X) - \dim(Y)$.*

(c) *If $\dim(Y) = \dim(X)$ then $K(X)$ is a finite algebraic extension of $K(Y)$. Assume also that $K(X)$ is separable over $K(Y)$. Then there is a dense open set $U \subseteq Y$ such that for all $u \in U$, the fiber $g^{-1}(u)$ is a finite set of cardinality $d = [K(X) : K(Y)]$.*

Proof. Given any three fields $K \subseteq \mathcal{F} \subseteq \mathcal{G}$ the transcendence degree of $\mathcal{G}$ over K is the sum of the transcendence degree of $\mathcal{F}$ over K and the transcendence degree of $\mathcal{G}$ over $\mathcal{F}$. Part (a) follows from applying this to $K \subseteq K(Y) \subseteq K(X)$ along with the theorem that the dimension of a variety over K is the transcendence degree of its function field over K.

To prove part (b), let $R = K[Y] \subseteq K[X] = S$. Then S is a domain finitely generated over the domain R, and by the Noether normalization theorem for domains, we may localize at one nonzero element $f \in R$ so that S_f is a module-finite extension of a polynomial ring over R. The number of variables must be δ, the transcendence degree. Let U be the open set corresponding to $D(f)$ in Y. Thus, after replacing R and S by R_f and S_f, it suffices to show that if S is a module-finite domain extension of $R[x_1,\, \ldots,\, x_\delta]$, then all fibers over maximal ideals m of R have dimension δ. Since S/mS is module-finite over $(R/m)[x_1,\, \ldots,\, x_\delta]$ the dimension is at most δ. Since S has prime ideal Q lying over

$mR[x_1, \ldots, x_\delta]$ by the lying over theorem, and we have $S/mS \twoheadrightarrow S/Q$, while S/Q is a module-finite extension domain of $(R/m)]x_1, \ldots, x_\delta]$, we also have that the dimension is at least δ.

It remains to consider part (c). We continue the notations from the proof of (b). The first statement is immediate from (a) and the fact that $S = K[X]$ is finitely generated over K and, hence, over $R = K[Y]$. We may localize at $f \in R - \{0\}$ and so assume that S is module-finite over R. Then $K(Y) \otimes_R S = K(X)$. Choose a primitive element θ for $K(X)$ over $K(Y)$: by multiplying by a suitable nonzero element in R, we may assume that θ is in S. Let G be the minimal monic polynomial of θ over the fraction field of R. By our hypothesis on the field extension, G will be separable over R. By inverting one more element of $R - \{0\}$ we may assume that the coefficients of G are in R. Note that $S/R[\theta]$, as an R-module, is torsion. Therefore we may invert yet another element of R and assume without loss of generality that $S = R[\theta]$, and then $S \cong R[x]/G$.

Consider the roots of G in a suitably large extension field of the fraction field of R. The product of the squares of their differences (the discriminant of G) is a symmetric polynomial over $\mathbb{Z}$ in the roots of G, and therefore is expressible as a polynomial D over $\mathbb{Z}$ in the coefficients of G, which, up to sign, are the elementary symmetric functions of the roots. The discriminant is therefore a nonzero element of R. We localize at the discriminant as well, and so we may assume that it is a unit of R. Note that each localization has the effect of restricting out attention to a smaller dense open subset of Y.

The points of the fiber over a m, a maximal ideal of R, correspond to the maximal ideals of $(R/m)[x]/\overline{G}$, where $\overline{G}$ is simply the image of G modulo m. But $R/m = K$ and the discriminant of $\overline{G}$ is simply the image of the discriminant of G (one substitutes the images of the coefficients of G into D), and so is not zero. It follows that the roots of G are mutually distinct, and so the number of points in the finite fiber is precisely the degree of G, which is the same as the degree of G and is equal to $[K(Y) : K(X)]$. □

Part (b) of the preceding remark shows that for a dominant of map of varieities $X \to Y$, the dimension of X is the same as the sum of the dimension of Y and the dimension of a "typical" fiber over a point of Y, in the sense that this is true for all points of a dense Zariski open subset of Y contained in the image of X.

We are now ready to show the irreducibility of the algebraic sets corresponding to the ideals we are claiming to be prime.

Proposition. *With notation as in the Theorem, if σ is standard, $V = V(I_\sigma(X) + J_{s_k}(X))$ is irreducible.*

Proof. Consider $r \times (t-1)$ matrices B such that the first k entries of the first row are 0. These may be thought of as the points of $\mathbb{A}_K^{r(t-1)-k}$. Let C_j be a $j \times (s_j - s_{j-1})$ matrix over K, $1 \le j \le t-1$. (Recall that $s_0 = 0$ and $s_{t-1} = s$.) Consider the matrix

$$A = B|_1 C_1 \# B|_2 C_2 \# \cdots \# B|_{t-1} C_{t-1}.$$

The first k columns are in the span of the columns of $B|_k$ and so all have a 0 as their initial entry. Moreover, the columns of $A|_{s_j}$ are in the span of the columns of $B|_j$ for every j, and so the rank of $A|_{s_j}$ is at most j for every j. That is, A is a point of V. The choices for C_j are parametrized in bijective fashion by the points of $\mathbb{A}_K^{j(s_j-s_{j-1})}$ for all j. Therefore, we have a map $\mathbb{A}_K^N \to V$, where

$$N = r(t-1) - k + \sum_{j=1}^{t-1} j(s_j - s_{j-1}).$$

To show that V is irreducible, it suffices to show that this map is onto.

Consider any matrix A representing a point of V. Let V_j be the span of the columns of $A|_{s_j}$. Then the V_j satisfy the conditions of the Lemma, and we may choose W_j as in the lemma: the linear functional is projection on the first coordinate. Choose B so that its first column spans W_1, its first two columns span W_2, and, in general, its first j columns span W_j. It is a straightforward induction to prove that this can be done.

Now the columns of $A|_{s_j}$ are in the span of the columns of $B|_j$ for all j: in particular, this is true for the last $s_j - s_{j-1}$ columns, which says precisely that the matrix formed from those columns has the form $B|_j C_j$, as required. $\square$

We can now compute the dimension of V, keeping the above notation. We can consider the open set $U \subseteq V$ where the matrix formed by the columns indexed by the $s_{j-1}+1$, $1 \le j \le t-1$, has rank $t-1$: call this matrix B. Note that U is non-empty because we can use part of the standard basis $e_2, \ldots, e_t$ for K^r for the columns of A indexed by the numbers $s_{j-1}+1$, $1 \le j \le t-1$, and take the rest of the columns of A to be 0.

For each j, the submatrix D_j of A consisting of the columns indexed by $s_{j-1}+1, \ldots, s_j$ can be written uniquely as a linear combination of the columns of $B|_j$. The coefficients needed comprise the columns of a $j \times (s_j - s_{j-1})$ matrix C_j. Note that the first column of C_j is the last column vector in the standard basis for K^j: this corresponds to the fact that the first column of D_j is the same as the column of A indexed by $s_{j-1}+1$ and is the j th column of B. It is therefore the last column of $B|_j$. The entries of C_j other than the first column are arbitrary scalars and therefore C_j may be thought of as varying in an affine space $A^{j(s_j-s_{j-1}-1)}$, and this is also true, therefore, of D_j. It follows that the dimension of V should be

$$r(t-1) - k + 1(s_1 - 1) + 2(s_2 - s_1 - 1) + \cdots + (t-1)(s_{t-1} - s_{t-2} - 1)$$

which we can rewrite as

$$r(t-1) - k - (s_1 + s_2 + \cdots + s_{t-2}) + (t-1)s - \binom{t}{2}$$

.

We can make this more precise as follows. Let $W \subseteq \mathbb{A}_K^{r(t-1)}$ be the non-empty open set consisting of matrices of rank $t-1$, and let $f : U \to W$ be the map that sends the matrix A to the matrix $B = f(A)$ consisting of the columns of A with indices $s_{j-1}+1$, $1 \leq j \leq t-1$. For fixed B, consider the fiber of f over B. Let V_j be the vector space spanned by the first j columns of B. Then the fiber may be described as consisting of all matrices A such that each column of A indexed by $s_{j-1}+1$, $1 \leq j \leq t-1$, is the j th column of B and each column of A with index h, $s_{j-1}+1 < h \leq s_j$ is in the vector space V_j. It follows that the fiber is isomorphic with

$$\prod_{j=1}^{t-1} V_j^{s_j - s_{j-1} - 1},$$

so that each fiber has dimension $d = \sum_{j=1}^{t-1} j(s_j - s_{j-1} - 1)$, and so the dimension of U (and, likewise, of V) is the sum of the dimensions of W and d, as required. We have now proved:

Theorem. *With notation as above, if σ is standard then*

$$\dim\big(V(I_\sigma(X) + J_{s_k}(X))\big) = (r+s)(t-1) - k - (\sum_{j=1}^{t-2} s_j) - \binom{t}{2}. \quad \square$$

We can now complete the proof that all the ideals of the form $I = I_\sigma(X) + J_a(X)$ are radical. Because of our result on irreducibility, this also shows that the ones where $a = s_k$ are prime. As usual we may assume $a < s$ or else we can work with the matrix obtained by deleting the first row of X instead. Let $x = x_{1,a+1}$. We use the two lemmas that are the basis for the method of principal radical systems. If we specialize x to 1 and all other entries of the matrix to 0 we see that we have a point A where all generators of I vanish but x does not. Thus, $I + (x) = I_\sigma + J_{a+1}(X)$ is strictly larger than I, and therefore radical by the induction hypothesis. If $a = s_k$ for some k we are done, since we know that $\mathrm{Rad}\,(I)$ is then prime. Otherwise we have that $s_k < a < s_{k+1}$ for some k. In this case, from the lemma on killing minors we have that $xI_k(X|_a) \subseteq I$. Let σ' be the t-tuple that agrees with σ except that we change the $k+1$ st entry s_k to a. By the induction hypothesis, $I_{\sigma'}(X) + J_a(X)$ is radical and, therefore, prime: call it P, and $I \subseteq P$. But $xP \subseteq I$, and so I is radical. $\square$

Our next objective is to prove the Cohen-Macaulayness assertions in the statement of the second the Theorem on p. 215 (the seventh page of the notes for this lecture). The argument is entirely similar to what we did earlier in studying the ideal generated by the 2×2 minors of a matrix of indeterminates.

We use reverse induction, assuming the result that larger ideals of the form $I_\sigma + J_{s_k}(X)$ are Cohen-Macaulay.

Suppose that a specific prime of the form $I_\sigma + J_{s_k}(X)$ is given. Call the ideal P. To show that $K[X]/P$ is Cohen-Macaulay, it suffices to show that the depth of $K[X]/P$ on the ideal m generated by all the $x_{i,j}$ in $K[X]$ is $d = \dim(K[X]/P)$. REF EARLIER Let $x = x_{1,s_k+1}$. Since we already know that $K[X]/P$ is a domain, we have that x is a not a zerodivisor, and so $K[X]/P$ is Cohen-Macaulay if and only if $K[X]/(P + xK[x])$ is, and this may be described as $K[X]/(I_\sigma(X) + J_{s_k+1}(X))$.

There are two cases. If $s_k+1 = s_{k+1}$, then $I+xK[X]$ is a larger prime ideal of our family, and so killing it gives a Cohen-Macaulay ring by the induction hypothesis. If $s_k+1 < s_{k+1}$ then $I + (x)$ is radical. By the lemma on killing minors, each of the variables $x_{1,b}$ for $s_k+1 < b < s_{k+1}$ kills $I_k(X|_{s_k+1})$. Let σ' be the result of changing s_k in σ to s_k+1, while leaving all other entries fixed. Let $Q_1 = I_{\sigma'} + J_{s_k+1}(X)$ and $Q_2 = I_\sigma(X) + J_{s_{k+1}}(X)$. Both of these ideals are prime, and we know that they have Cohen-Macaulay quotients by the induction hypothesis. This is also true for $Q_3 = Q_1 + Q_2 = I_{\sigma'}(X) + J_{s_{k+1}}(X)$.

Note that $V(P + (x)) = V(Q_1) \cup V(Q_2)$ by the lemma on killing minors: since all of the ideals are radical, we have that $P = Q_1 \cap Q_2$.

Moreover, $K[X]/Q_1$ has dimension $d-1$: among the numbers used in calculating the dimension, s_k has increased by one while all others, including k, have not changed. Similarly, $K[X]/Q_2$ has dimension mat$d-1$: here, only k has changed, increasing by 1. Finally, $K[X]/Q_3$ has dimension $d-2$, since in this case k has increased by 1 and s_k has increased by one. Since these are Cohen-Macaulay, in the short exact sequence

$$0 \to K[X]/(P + (x)) \to K[X/Q_1 \oplus K[X]/Q_2 \to K[X]/Q_3 \to 0$$

the depths of the middle and right hand terms on m are $d-1$ and $d-2$ respectively, and so the depth of $K[X]/(P + (x))$ is $d-1$, as required. □

Corollary. *For any field K and $r \times s$ matrix of indeterminates X, if $! \le t \le r \le s$ then $R = K[X]/I_t(X)$ is a Cohen-Macaulay normal domain.*

Proof. We have already established that R is a Cohen-Macaulay domain. What is left to prove is normality. We use induction on t. If $t = 1$, the quotient is a field and there is nothing to prove. Hence, we may assume $t \ge 2$. The Cohen-Macaulay condition implies that all associated primes of ideals generated by regular sequences $f_1, \ldots, f_h$ have height h. In particular, since this is true when $h = 1$, we need only show that the localization of R at any height one prime P is Noetherian discrete valuation ring. P cannot contain all the indeterminates, and, by symmetry we, may assume that $x_{11} \notin P$. Then R_P is also a local ring at a height one prime of the ring $R[1/x_{11}]$. Thus, it suffices to show that $R[1/x_{11}]$ is normal. Multiply the first row of the matrix X by $1/x_{11}$. The upper left entry is 1. Now perform elementary column operations to make the other entries of the first row 0, and then elementary row operations to make the other entries of the first column zero. These operations do not affect the ideal generated by the size t minors. The new matrix is the direct sum of a 1×1 block, $\big(1\big)$, and an $(r-1) \times (s-1)$ matrix X' whose typical entry is $x'_{ij} = x_{ij} - (x_{i1}x_{1j}/x_{11})$, $2 \le i \le r$, $2 \le j \le s$. It is easy to see that the

$K[X][1/x_{11}] = K[X'][x_{ij} : i = 1 \text{ or } j = 1][1/x_{11}]$, which implies that the entries of X' and the indeterminates in the first row and first column of X are algebraically independent. It is also easy to check that the ideal generated by the t size minors of X becomes the ideal generated by the $t-1$ size minors of X'. It follows that $R[1/x_{11}]$ is a localization of a polynomial ring over $K[X']/I_{t-1}(X')$, and it follows from the induction hypothesis that $K[X']/I_{t-1}(X')$ is normal. $\square$