Math 615, Winter 2020 **Problem Set #1**
Due: Wednesday, February 5

1. Let $(R, m\,K)$ be a local ring. If $f \in R - \{0\}$, there exists a unique $k \in \mathbb{N}$ such that $f \in m^k - m^{k+1}$. Then the image $\mathcal{L}(f)$ of f in $m^k/m^{k+1} = [\mathrm{gr}_m(R)]_k$ is called the *leading form* of f. Show that if the leading form of f is a nonzerodivisor in $\mathrm{gr}_m(R)$, then f is a nonzerodivisor in R.

2. Let M be an R-module and f, g a regular sequence on M. Prove that $\mathrm{Ann}_M g \subseteq f\mathrm{Ann}_M g$ and that f is a nonzerodivisor on M/gM. Conclude that if $(*)$ (R, m) is local with $f, g \in m$) (respectively, $(**)$ $\mathbb{N}$-graded, with f, g forms of degree > 0) and M is finitely generated (respectively, $M_{-n} = 0$ for all $n \gg 0$), then g, f is a regular sequence on M. [Transpositions of consecutive elements generate the symmetric group S_n: hence, when $(*)$ or $(**)$ holds, finite regular sequences (of forms in the graded case) are permutable.]

3. Let $R = K[[x_1, \ldots, x_n]]$ be a formal power series ring over a field K and I a proper ideal. You may assume that $\mathrm{gr}_m(R/I)$ is $K[X_1, \ldots, X_n]/J$, where J is the ideal generated by all lowest degree forms (called *leading forms*) of elements of $I-\{0\}$. Show that if $I = fR$, then J is generated by the leading form of f. Show that in $R = K[[x_1, x_2]]$, if $I = (f, g)R$, where $f = x_1^2$ and $g = x_1x_2 + x_2^3$, J needs more than two generators.

4. If R is $\mathbb{N}$-graded, the kth Veronese subring $R^{(k)}$ of R is defined as the $\mathbb{N}$-graded algebra whose nth graded component is R_{kn}. Thus, $R^{(k)} = \bigoplus_{n=0}^{\infty} R_{kn}$. Let R be a standard graded K-algebra, where K is a field, $\dim(R) = d$, and R has mnultiplicity e. Use a Hilbert function argument to determine the multiplicity of $R^{(k)}$.

5. If R, S, are finitely generated $\mathbb{N}$-graded algebras over A, the tensor product $T = R \otimes_A S$ is $(\mathbb{N} \times \mathbb{N})$-graded by $T_{ij} = R_i \otimes_A S_j$, and the *Segre product* $R\#_A S := \bigoplus_{i=0}^{\infty} R_i \otimes S_i \subseteq T$ is $\mathbb{N}$-graded with $[R\#_A S]_i = R_i \otimes_A S_i$. Suppose that R, S are standard graded algebras over a field K with Krull dimensions d_R, d_S and multiplicities e_R, e_S (with respect to the homogeneous maximal ideals). Use a Hilbert function or Poincaré series argument to determine the multiplicities of $R \otimes_K S$ and $R\#_K S$. (You may assume these are standard.)

6. (a) Consider an $r \times s$ matrix of indeterminates $X = \big(x_{ij}\big)$ over a field K, where $2 \le r \le s$, and let R be the polynomial ring $K[X] := K[x_{ij} : 1 \le i \le r, 1 \le j \le s]$ in rs variables over K with the usual grading. Let I be the ideal generated by the size 2 minors of X. Assume the result of Problem **EXTRA CREDIT 2** below: that the K-algebra map $K[X] \twoheadrightarrow S$, where $S = K[y_1, \ldots, y_r, z_1, \ldots, z_s]$, such that $x_{ij} \mapsto y_i z_j$ has kernel I, so that $K[X]/I \cong K[y_1, \ldots, y_r]\#_K K[z_1, \ldots, z_s]$. Thus, you may assume that the Krull dimensions of $K[X/I]$ and S are $r + s - 1$. What is the multiplicity of $K[X]/I$?
(b) Now assume $r = 2$. Prove that the $s+1$ elements x_{21}, $x_{1,j} - x_{2,j+1}$, for $1 \le j \le s-1$, and $x_{1,s}$ form a (linear) homogeneous system of parameters for $K[X]/I$. Prove that the quotient A of $R[X]/I$ by the ideal this homogenous system of parameters generates is isomorphic with $K[x_{1,1}, \ldots, x_{1,s-1}]/\mathcal{M}^2$ where $\mathcal{M} = (x_{1,1}, \ldots, x_{1,s-1})$. How does $\ell(A)$ compare with the multiplicity of $K[X]/I$?

EXTRA CREDIT 1. Let R be a finitely generated $\mathbb{N}$-graded algebra over $A = R_0$. Show that for some integer $k > 0$, the Veronese subring $R^{(k)}$ is a standard graded A-algebra.

[This implies that if A is Artin local and M is a finitely generated graded R-module then there are polynomial functions $H_0, \ldots, H_{k-1}$ such that for all $n \gg 0$ and for all i, $0 \leq i \leq k-1$, $\dim_K(M_{kn+i}) = H_i(n)$.]

EXTRA CREDIT 2. With hypothesis as in Problem **6.** part (a), exhibit a family $\mathcal{B}$ of monomials in $K[X]$ whose images in S are in bijective correspondence with the monomials spanning S, and show that modulo I, every monomial in $K[X]$ is congruent to a monomial in $\mathcal{B}$. Explain why the existence of $\mathcal{B}$ proves that $K[X]/I \cong S$.