Math 615, Fall 2020 **Problem Set #1: Solutions**

1. If $fg = 0$ with $g \neq 0$ and $g \in m^s - m^{s+1}$, then $\mathcal{L}(f)\mathcal{L}(g) \in [\mathrm{gr}_m(R)]_k[\mathrm{gr}_m(R)]_s \subseteq [\mathrm{gr}_m(R)]_{k+s}$ is im $fg = 0$ in m^{k+s}/m^{k+s+1}, so that $\mathcal{L}(f)\mathcal{L}(g) = 0$, a contradiction. $\square$

2. $u \in \mathrm{Ann}_M g$ then $ug \equiv 0 \in M/fM$, and so $ug = fv$ with $v \in M$. Since g is not a zerodivisor on M/fM, we have $u = fw$ with $w \in M$. Then $gu = 0 \Rightarrow gfw = 0 \Rightarrow gw = 0$ since f is not a zerodivisor on M. But then $u = fw$ with $w = \mathrm{Ann}_M g$, as required. When Nakayama's lemma holds, this implies $\mathrm{Ann}_M g = 0$, and so g is a nonzerodivisor on M. For the remaining part, if f is a zerodivisor on M/gM, say $fu = gv$, then since f, g is a regular sequence, $v \in fM$, say $v = fw$. Then $fu = gfw \Rightarrow f(u - gw) = 0 \Rightarrow u - gw = 0$, since f is not a zerodivisor on M. But then the image of u is 0 in M/gM. $\square$

3. J is generated by the leading form of f because in the power series ring, the lowest degree term of fg is the product of the lowest degree terms, and so $\mathcal{L}(fg) = \mathcal{L}(f)\mathcal{L}(g)$. For the second part, it is clear clear that x_1^2 and x_1x_2 are in J and among the minimal generators of J (which can only contain elements of degree 2 or more), but more generators are needed since $\mathcal{L}((-x_1 + x_2^2)g + x_2 f) = x_2^5$.

4. If the Hilbert function of R has leading term $\frac{e}{(d-1)!}n^{d-1}$, the leading term for the Veronese subring $R^{(k)}$ will be $\frac{e}{(d-1)!}(kn)^{d-1}$, and so the multiplicity of $R^{(k)}$ will be $k^{d-1}e$.

5. Note that the Poincaré series of a standard graded algebra of Krull dimension d has the form $P(z)/(1-z)^d$. If we rewrite $P(z) = Q(1-z)$, where Q has constant term c, this becomes $c/(1-z)^d +$ terms with a lower power of $(1-z)$ in the denominator. When we calculate the Hilbert function, only the $c/(1-z)^d$ contributes a polynomial of degree $d-1$: the other terms contribute polynomials of lower degree. So the coefficient of z^{d-1} in the Hilbert polynomial is $c/(d-1)!$ and the multiplicity is $c = Q(0) = P(1)$. In the case of a tensor product, the Poincaré series of the tensor product is the product of the two Poincaré series, $P_R(z)/(1-z)^{d_R}$ and $P_S(z)/(1-z)^{d_S}$, and the Krull dimension of the tensor product is $d_R + d_S$. Hence the multiplicity of the tensor product is $P_R(1)P_S(1) = e_R e_S$.

In the case of a Segre product, the Hilbert function is the product of the Hilbert functions, and the leading term is $\big(e_R n^{d_R-1}/(d_R-1)!\big)\big(e_S n^{d_S-1}/(d_S-1)!\big)$. Hence, the dimension is $(d_R - 1) + (d_S - 1) + 1 = d_R + s_S - 1$, and the multiplicity is
$(d_R + d_S - 2)! e_R e_S / \big((d_R - 1)!(d_S - 1)!\big) = \binom{d_R + d_S - 2}{d_R - 1} e_R e_S$.

6. (a) It is immediate from **5.** and **Extra Credit 2.** that the multiplicity is $\binom{r+s-2}{r-1}$.

(b) The specified elements have the correct cardinality, since the dimension in this case is $2+s-1 = s+1$, and so it suffices to show that the quotient is as specified. Modulo the specified elements the matrix has the form $\begin{pmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1,s-2} & x_{1,s-1} & 0 \\ 0 & x_{11} & x_{12} & \dots & x_{1,s-3} & x_{1,s-2} & x_{1,s-1} \end{pmatrix}$
The minors involving the first column give the products $x_{11}x_{1,j}$, for $1 \leq j \leq s-1$. Use induction on i: once we have all the products $x_{1,a}x_{1,j}$ for $a \leq i < s-1$, $1 \leq j \leq s-1$, we get the product $x_{1,i+1}x_{1,j}$ for $j \geq i+1$ (we already have the products for $j \leq i$)) using the minor involving the $(i+1)$ st and $(j+1)$ st columns: the main diagonal term in the minor is the product we want, and we already have the other term. The quotient has as K-vector space basis $1, x_{11}, x_{12}, \dots, x_{1,s-1}$, so the length of the quotient is s, which is the same as the multiplicity $\binom{2+s-2}{2-1} = s$