Math 615, Winter 2020 **Problem Set #2**
Due: Wednesday, February 19

1. Let M and N be modules over a ring R, let $x \in R$ be in the annihilator of N, and suppose that x is not a zerodivisor on M. Show that for all i, there is an exact sequence

$$0 \to \mathrm{Tor}_i(M, N) \to \mathrm{Tor}_i(M/xM,\, N) \to \mathrm{Tor}_{i-1}(M,\, N) \to 0.$$

Hence, $\mathrm{Tor}_i(M/xM, N) = 0$ if and only if $\mathrm{Tor}_i(M, N)$ and $\mathrm{Tor}_{i-1}(M, N)$ are both 0.

2. Let M and N be modules over a ring R, let $x_1, \ldots, x_d \in R$ be in the annihilator of N, and suppose that $x_1, \ldots, x_d$ is an improper regular sequence on M. Show that $\mathrm{Tor}_i(M, N) = 0$ for $i \geq n$ if and only if $\mathrm{Tor}_i(M/(x_1, \ldots, x_d)M,\, N) = 0$ for $i \geq n + d$.

3. Let $0 \to A \to B \to C \to 0$ be an exact sequence over the ring R, let $f \in R - \{0\}$, let M be an R-module, and suppose that either M_f or C_f is R-flat. Show that if f is not a zerodivisor on $A \otimes_R M$, then $A \otimes_R M \to B \otimes_R M$ is injective.

4. Let (R, m, K) be the local ring $K[[x, y]]/(x^2, xy)$. Give a recursive description of the minimal modules of syzygies arising in a minimal free resolution $F_\bullet$ of $K = R/m$ over R. Also give a recursive description of the sequence of ranks of the free modules in $F_\bullet$, which are the same as the K-vector space dimensions of the modules $\mathrm{Tor}_n^R(K, K)$.

5. Let R be a local ring and let M be a finitely generated R-module of projective dimension n. Let $R \to S$ be a local homomorphism (so that the maximal ideal of R maps into the maximal ideal of S). Suppose that $\mathrm{Tor}_i^R(M,\, S) = 0$ for $i \geq 1$. (This holds if S is R-flat or if $S = R/(x_1, \ldots, x_d)$ where $x_1, \ldots, x_d$ is a regular sequence on both R and M.) Show that the projective dimension of $S \otimes_R M$ over S is also n.

6. Let (T, m) be a local ring of Krull dimension $d + 1$ and let f be a nonzerodivisor in m. Let $R = T/fR$. Let M be a finitely generated nonzero R-module that has finite projective dimension over T (always true if T is regular). Show that a dth module of syzgies Q of M over R is 0 or else has projective dimension 1 over T, with Betti numbers equal.

Extra Credit 3. Suppose that T, f, R and Q are as in the hypothesis for Problem **6.** just above, and let $0 \to T^n \xrightarrow{\alpha} T^n \to Q \to 0$ be a minimal free resolution of Q over T, where α is an $n \times n$ square matrix over T. Let $e_1, \ldots, e_n$ be the standard basis for T_n. Then fe_j is in the column space of α for every j (explain why), and can be written αv_j for some $n \times 1$ column v_j. Let I_n be the size n identity matrix over T. Show that $fI_n = \alpha\beta$, where the columns of β are the v_j. Explain why $\beta\alpha = fI_n$ as well. Prove that

$$\cdots \xrightarrow{\beta} R^n \xrightarrow{\alpha} R^n \xrightarrow{\beta} R^n \xrightarrow{\alpha} R^n \to Q \to 0$$

gives a free resolution of Q over R.

EXTRA CREDIT 4. Let x be a variable over the rational numbers and let R be the subring of $\mathbb{Q}[x]$ consisting of polynomials all of whose values on integers are integers. The polynomial $C_n(x) = \frac{1}{n!}x(x-1)\cdots(x-n+1)$ is an example for every $n \geq 0$ (by convention, $C_0(x)$ is the constant function 1). Is R Noetherian? Prove your answer.