Math 615, Fall 2020 **Problem Set #2: Solutions**

1. The long exact sequence for Tor coming from $0 \to M \xrightarrow{x\cdot} M \to M/xM \to 0$ by applying $\otimes_R N$ yields $\mathrm{Tor}_i(M,N) \xrightarrow{x\cdot} \mathrm{Tor}_i(M,N) \to \mathrm{Tor}_i(M/xM,\, N) \to \mathrm{Tor}_{i-1}(M,\, N) \xrightarrow{x\cdot} \mathrm{Tor}_{i-1}(M,\, N)$. Since x kills N, it kills all the $\mathrm{Tor}_j(M,N)$, which implies that the first and last displayed maps are 0 and so have image 0, and this yields the required exact sequences at once. The final assertion is then obvious. □

2. The case $d = 0$ is obvious, the case $d = 1$ is immediate from Problem **1.**, and the general case follows from the case $d = 1$ by a straightforward induction on d. □

3. The hypothesis leads to the conclusion that $\mathrm{Tor}_i(M_f,\, C) = 0$ or $\mathrm{Tor}_i(M,\, C_f) = 0$, $i \geq 1$, and both are isomorphic with $\mathrm{Tor}_i(M,\, C)_f$. It follows that every element of $\mathrm{Tor}_i(M,\, C)$ is killed by a power of f for $i \geq 1$. In particular, this holds for $\mathrm{Tor}_1(M,\, C)$, which is the only case we need. From the long exact sequence for Tor, we have $\mathrm{Tor}_1(M,C) \to M \otimes_R A \to M \otimes_R B$ is exact, and since f, and, hence, its powers are nonzerodivisors on $M \otimes_R A$, the image of $\mathrm{Tor}_1(M,C)$ in $M \otimes_R C$ must be 0, and the map $N \otimes_R A \to M \otimes_R B$ is injective. □

4. Clearly, $\mathrm{syz}^1 K = m = (x,y)R$. Note $Kx = Rx \cong R/m = K$, and that every element of R can be written uniquely in the form $cx + f$ where $c \in K$ and $f \in K[y]$. y kills such an element if and only if $f = 0$. Map $R^2 \twoheadrightarrow m$ so that $(r,s) \mapsto rx + sy$, and let N be the kernel. Since $xm = 0$, the kernel N contains $m = m \oplus 0 \subseteq R \oplus R$.. Let $(r,s) \in N$. If r is unit, we would have $x = -r^{-1}sy$, which is impossible since $x \notin yR$. It follows that $(r,s) \in N$ if and only if $r \in m$ and $sy = 0$, which implies that $s \in xR \cong R/m = K$. Thus, $\mathrm{syz}^2 K = \mathrm{syz}^1 m \cong m \oplus K$. Since $\mathrm{syz}^1(A \oplus B) \cong \mathrm{syz}^1(A) \oplus \mathrm{syz}^1(B)$, we have by induction that $\mathrm{syz}^n(K) \cong K^{\oplus f_{n-1}} \oplus m^{\oplus f_n}$ for all $n \geq 1$, where the f_n are the Fibonacci sequence, given recursively by $f_0 = 1$, $f_1 = 1$, and $f_{n+1} = f_n + f_{n-1}$, $n \geq 1$. The rank of the n th free modules in a minimal resolution is the same as the same as the least number of generators of an n th module of syzygies, which is $f_{n-1} + 2f_n = (f_{n-1} + f_n) + f_n = f_{n+1} + f_n = f_{n+2}$. □

5. The projective dimension of M over R is the same as the length d of a minimal free resolution $0 \to G_d \to \cdots \to G_0 \to 0$ of M (M is the the cockerel of $G_1 \to G_0$. Consider the complex $0 \to S \otimes_R G_d \to \cdots \to S \otimes_R G_0 \to 0$ obtained by applying $S \otimes_R \_$. The homology of this complex for $i \geq 1$ is $\mathrm{Tor}_i^R(M,S) = 0$ by hypothesis. Therefore, this complex is acyclic, and is a free resolution of its zeroth homology module, which is $S \otimes_R M$. Since the maximal ideal of R maps into the maximal ideal of S, the matrices of the maps of $S \otimes_R G_\bullet$ have entries in the maximal ideal of S, and so this is a minimal free resolution, and the projective dimension of $S \otimes_R M$ over S is therefore also d.

6. Neither M nor any nonzero module of syzygies of M can have projective dimension 0, for that would mean the module is free and is not killed by f. If $\mathrm{pd}_T M > 1$, then since $pd_T(T/fT)^{\oplus h} = 1$, $\mathrm{syz}^1(M)$ projective dimension one smaller than that of M. If $\mathrm{pd}_T M = 1$, a nonzero module of syzygies over T/fT is 0 or has projective dimension 1 over T. The projective dimension of M is over T is at most the depth of T on its maximal ideal, which is at most d. Hence, a nonzero d th module of syzygies N over T/fT must have projective dimension 1 over T. Let its minimal resolution over T be

$0 \to T^h \to T^n \to N \to 0$. If we localize at f we have that $N_f = 0$ and so $T_f^h \cong T_f^n$. Tensoring with $K = T/m$, we have $K^h \cong K^n$, and so $h = n$ as required.