Math 615, Winter 2020 **Problem Set #3**

Due: March 11

1. Let $K[s,t]$ be polynomial over the field K, and let R be the local ring of $K[s^4, s^3t, st^3, t^4] \subseteq K[s,t]$ at the graded maximal ideal. Determine the Koszul homology modules of R with respect to the system of parameters $x = s^4$, $y = t^4$, including their lengths. Determine $\chi(x, y;\, R)$. Is R Cohen-Macaulay?

2. Let $(R, \mathfrak{m}, K)$ be local and M a Cohen-Macaulay R-module of Krull dimension d. Show that if $x = x_1$ is a nonzerodivisor on M, then $\mathrm{Ext}^d_R(K, M) \cong \mathrm{Ext}^{d-1}_R(K, M/xM)$. [It follows at once that for any regular sequence $x_1, \ldots, x_h \in \mathfrak{m}$ on M, $\mathrm{Ext}^d_R(K, M) \cong \mathrm{Ext}^{d-h}_R(K,\, M/(x_1, \ldots, x_h)M)$ and that for a system of parameters $x_1, \ldots, x_d$ for M, $\mathrm{Ext}^d_R(K, M) \cong \mathrm{Hom}_R\big(K, M/(x_1, \ldots, x_d)M\big)$. The dimension of this K-vector space is called the *(Cohen-Macaulay) type* of M.]

3. Let $M \neq 0$ be a finitely generated module over a local ring $(R, \mathfrak{m}, K)$ and suppose $\mathrm{depth}_{\mathfrak{m}} M = h$. Show that every nonzero submodule of M has Krull dimension at least h. [Suggestion: use induction on h. If $h > 0$ show that there is a maximum submodule $N \subseteq M$, of dimension $< h$ and construct a nonzerodivisor $x \in \mathfrak{m}$ on M, and M/N (and, hence, N). Consider $N/xN \hookrightarrow M/xM$.]

4. Let M be an R-module. If $\underline{x} = x_1, \ldots, x_n \in R$, $y \in R$, and $z \in R$, show there is a long exact sequence $\cdots \to H_i(\underline{x},\, y; M) \to H_i(\underline{x},\, yz; M) \to H_i(\underline{x},\, z; M) \to \cdots$. Deduce that if R is local, M is finitely generated, and all three are defined, then $\chi(\underline{x},\, yz; M) = \chi(\underline{x},\, y; M) + \chi(\underline{x},\, z; M)$.

5. Let $(A, \mathfrak{m})$ be an Artin local ring, and $x_1, x_2 \in \mathfrak{m}$. Does $H_1(x_1,\, x_2;\, A)$ need at least two generators? Prove your answer.

6. Let R be a finitely generated standard $\mathbb{N}$-graded algebra over a field K (so that $R_0 = K$ and R is generated by its 1-forms). Let $\mathfrak{m}$ be the homogeneous maximal ideal in R. Assume that $x_1, \ldots, x_d \in [R]_1$ is a homogeneous system of parameters, where $d \geq 2$. Let $S = R\#_K K[y,z]$ be the Segre product of R with a polynomial ring in two variables over K. Note that $\dim(S)$ is $\dim(R) + 1$ by the solution to Problem **5.** of the first problem set.

(a) Show that S is isomorphic with the Rees ring $R[\mathfrak{m}t]$ of R with respect $\mathfrak{m}$.

(b) Show that $x_1z, x_1y - x_2z, \ldots, x_iy - x_{i+1}z, \ldots, x_{d-1}y - x_dz, x_dy$ is a homogeneous system of parameters for S.

EXTRA CREDIT 5. With notation as in Problem **6.**, show that if $R = K[x_1, \ldots, x_d]$ is polynomial, then S is Cohen-Macaulay. Also show that if $R = K[X_1, \ldots, X_n]/(F)$, where $X_1, \ldots, X_n$ are indeterminates, and F is monic as a polynomial in X_n of degree r (so that the images of $X_1, \ldots, X_{n-1}$ form a homogeneous system of parameters), and $r \geq n \geq 3$, then S is not Cohen-Macaulay.

EXTRA CREDIT 6. Let M be an R-module. Prove that M is flat over R is and only if for every finite set of elements $f_1, \ldots, f_n$ of R and relation $\sum_{j=1}^n f_i u_j = 0$ with elements $u_1, \ldots, u_n \in M$m there exist a finite set of relations on the f_j with coefficients in R, say $\sum_{j=1}^n r_{ij} f_j = 0$, $1 \leq j \leq h$, and elements $v_1, \ldots, v_h \in M$ such that $(u_1, \ldots, u_n) = \sum_{i=1}^h (r_{i1}v_i, \ldots, r_{in}v_i)$. (Roughly speaking, this says that a relation on $f_1, \ldots, f_n$ with coefficients in M is a consequence of relations on $f_1, \ldots, f_n$ with coefficients in R.)