Math 615, Winter 2020 **Problem Set #3: Solutions**

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1. $H_2(x, y; R) = 0$, since R is a domain and no nonzero element is killed by x, y. We may map R into the Veronese subring $S = K[s^i t^{4-i} : 0 \le i \le 4]$, which is a direct summand of $K[s,t]$: the complement is spanned over K by all monomials whose degree is not divisible by 4. Thus, S is normal, and, hence, Cohen-Macaulay. There is an exact sequence of R-modules $0 \to R \to S \to K \to 0$ where the copy of K is spanned by the image of x^2y^2. This yields a long exact sequence of Koszul homology $0 \to H_2(x, y; K) \to H_1(x, y; R) \to 0 \to H_1(X, y; K) \to H_0(x, y; R) \to H_0(x, y; S) \to H_0(x, y; K) \to 0$, Since $H_2(x, y; S) = H_1(x, y; S) = 0$. Thus, $H_1(x, y; R) \cong H_2(x, y; K) \cong K$, which shows that R is not Cohen-Macaulay. Also, $H_1(x, y; K) \cong K^2$ and $H_0(x, y; K) \cong K$, and so $\ell\big(H_0(x, y; R)\big) = \ell\big(H_0(x, y; S)\big) + 2 - 1 = \ell\big(H_0(x, y;\, S)\big) + 1$. Now, $H_0(x, y; S) \cong S/(s^4, t^4)$ is spanned by all monomials of total degree divisible by 4 but not divisible by s^4 or t^4: these monomials are $1, s^3t, s^2t^2, st^3$, since in total degree 8 or more, if we have s^it^j with $i + j \ge 8$, we must have $i \ge 4$ or $j \ge 4$. Hence, $\ell\big(H_0(x, y; R)\big) = 4 + 1 = 5$, and $\chi(x, y; R) = 5 - 1 + 0 = 4$. There are many other ways to do these calculations.

2. Since $0 \to M \xrightarrow{\cdot x} M \to M/xM \to 0$ is exact, the long exact sequence for $\mathrm{Ext}(K, \_\,)$ yields $\cdots \to 0 \to \mathrm{Ext}_R^{d-1}(K, M/xM) \to \mathrm{Ext}_R^d(K, M) \xrightarrow{\cdot x} \mathrm{Ext}_R^d(K, M)$ where the map induced by multiplication by x is 0 because x kills K and $\mathrm{Ext}_R^{d-1}(K, M) = 0$ (the leftmost term shown explicitly) because, since M is Cohen-Macaulay, it has depth d on the annihilator of K, which is $\mathfrak{m}$. The stated isomorphism is immediate. $\square$

3. If M has depth 0 on $\mathfrak{m}$ there is nothing to prove. Assume that $h \ge 1$ and use induction. Note that since a regular sequence on M in $\mathfrak{m}$ is part of a system of parameters on M, $\dim(M) \ge h$. Let N be a maximal submodule of M of dimension $< h$. Then $M/N \ne 0$. If $N' \subseteq M$ also has dimension $< h$, the fact that $N \oplus N'$ maps onto $N + N' \subseteq M$ shows that $N + N'$ has dimension $< h$ as well. Hence, $N\, incN$ and N is actually maximal among submodules of dimension $< h$. It follows that M/N has positive depth: if $\mathfrak{m} \in \mathrm{Ass}\,(M/N)$ and $u \in M - N$ has image $\overline{u}$ in M/N has image killed by $\mathfrak{m}$, then $0 \to N \to N + Ru \to R\overline{u} \to 0$ shows that $N + Ru$ also has dimension $< h$, since $R\overline{u} \cong K$. This contradicts the maximality of N. The union of the associated primes of M and N cannot cover $\mathfrak{m}$, since $\mathfrak{m}$ is not among these associated primes. Hence, we may choose $x \in \mathfrak{m}$ that is not a zerodivisor on M nor on M/N. Then $0 \to xN \to xM \to x(M/N) \to 0$ is an isomorphic subcomplex of $0 \to N \to M \to M/N \to 0$, and so it is exact. Killing it yields an exact sequence $0 \to N/xN \to M/xM \to (M/N)/x(M/N) \to 0$. Now M/xM has depth $h - 1$ on $\mathfrak{m}$, and its submodule N/xN has dimension $< h - 1$, contradicting the induction hypothesis.

4. Let $\underline{X} = X_1, \ldots, X_n$ and let $A = \mathbb{Z}[\underline{X}, Y, Z]$ be polynomial, which maps to R so that $\underline{X}, Y, Z$ are sent to $x_1, \ldots, x_n, y, z,$, respectively. There is a short exact sequence

$$0 \to A/(\underline{X}, Y) \xrightarrow{\cdot Z} A/(\underline{X}, YZ) \to A/(\underline{X}, Z) \to 0$$

which is the same as the short exact sequence

$$0 \to \mathbb{Z}[Y, Z]/(Y) \xrightarrow{\cdot Z} \mathbb{Z}[Y, Z]/(YZ) \to \mathbb{Z}[Y, Z]/(Z) \to 0.$$

The image of the leftmost nonzero module is identified with $Z\mathbb{Z}[Y, Z]/(YZ)$. M becomes an A-module by restriction of scalars, and we have a long exact sequence for Tor_A when we tensor with M. Since each denominator ideal is generated by a regular sequence in A, each quotient is resolved by a Koszul complex over A. Thus, the various $\mathrm{Tor}_i(\_\,; M)$ coincide with Koszul homology, and this yields the long exact sequence we want. In the situation where we are calculating χ, the formula we want is obtained by taking the alternating sum of all the lengths of terms in the (finite) long exact sequence we get, setting it equal to 0, and collecting terms. □

5. Yes, $H_1(x_1, x_2; A)$ needs at least two generators. Since the dimension of $A = 0$ and $2 > 0$, Serre's theorem yields that $\chi(x_1, x_2; A) = 0$, and so

$$\ell\big(H_1(x_1, x_2; A)\big) = \ell\big(H_0(x_1, x_2; A)\big) + \ell\big(H_2(x_1, x_2; A)\big).$$

The first term on the right is $\ell\big(A/(x_1, x_2)A\big)$, and the second term is nonzero, which shows that $\ell\big(H_1(x_1, x_2; A)\big) > \ell\big(A/(x_1, x_2)A)\big)$. But $H_1(x_1, x_2; A)$ is killed by $(x_1, x_2)A$, and so is a module over $A/(x_1, x_2)A$. If it had only one generator, its length would be $\leq \ell\big(A/(x_1, x_2)A)\big)$, a contradiction. □

6. (a) First note that if y is a new indeterminate, then there is an isomorphism of R with $K[Vy] \subseteq R[y]$ where V is the K-vector space R_1 of 1-forms of R. The space spanned over K by n-fold products of elements of Vy is precisely $R_n y^n$. The inverse isomorphism is the restriction of the R-algebra map $R[y] \to R$ that sends $y \mapsto 1$. Hence, the Rees ring $R[\mathfrak{m}t] \cong K[Vy, Vyt]$. Since y and yt are algebraically independent over R, this is isomorphic with $K[Vy, Vz]$, where $z = yt$ may be thought of as a new indeterminate, which is the same $R\#_K K[y, z]$.

(b) Since R is module-finite over its polynomial subring $A = K[x_1, \ldots, x_d]$, one has that $R\#_K[y, z]$ is module-finite over $K[x_1, \ldots, x_d]\#_K K[y, z]$. (If $g_1, \ldots, g_h$ is a homogeneous spanning set for R over A, then $R\#_K K[y, z]$ is spanned as a module over $A\#_K K[y, z]$ by finitely elements $g_i w_{ij}$, $1 \leq i \leq h$, where, for every i, the w_{ij} are the finitely many monomials in y, z of degree equal to that of g_i). The proposed homogeneous system of parameters has the right number of elements in it, since the dimension is $d+1$, and so it suffices to show that the elements are a homogeneous system of parameters in $K[x_1, \ldots, x_d]\#_K K[y, z]$. In fact, since the 2×2 minors of $\begin{pmatrix} x_1 y & x_2 y & \cdots x_j y & \cdots & x_d y \\ x_1 z & x_2 z & \cdots x_j z & \cdots & x_d z \end{pmatrix}$ vanish, the square of the ideal generated by the entries is 0 modulo the proposed parameters, from the result of Problem **6**, part (b) of the first problem set. □