Math 615, Winter 2020 **Problem Set #4**
Due: Wednesday, March 25

1. Let R be a ring and f, g nonzerodivisors.

Calculate (a) $\mathrm{Ext}^1(R/fR,\, R)$ and also (b) $\mathrm{Ext}^1(R/fR, R/gR)$ in these three cases:

(1) f, g is a regular sequence in R. (2) g is a multiple of f. (3) f is a multiple of g.

Use your results to discuss the behavior of $\mathrm{Ext}^1_R(M,\, N)$ where M and N are finitely generated modules over a principal ideal domain.

2. Let M be a finitely generated R-module over a regular local ring K. Show that $\dim_K(\mathrm{Ext}^i_R(M, K)$ is the rank of the i th free module in a minimal free resolution of M.

3. Let K be a field, let $S = K[X,\, Y,\, U,\, V]$ be polynomial, and let $R = S/(XU - YV) = K[x,\, y,\, u,\, v]$. Let $P = (x, y)R$ and $Q = (u, v)R$. Find a minimal free resolution of P and determine $\mathrm{Ext}^1_R\big(P, Q\big)$.

4. Let R be Noetherian and suppose that $f_1,\, \ldots\, , f_h \in R$ form a regular sequence on the finitely generated R-module M. Show that $[M/(f_1,\, \ldots\, , f_h)M] = 0$ in $G_0(R)$.

5. Let $S = K[x_1,\, \ldots\, , x_n]$ be a polynomial ring in n variables over a field K. Let $f = x_1 \cdots x_n$ be the product of the variables. Let $R = S/fS$. Prove that $G_0(R) \cong \mathbb{Z}^n$.

6. Let M be an R-module, where R is a regular local ring of Krull dimension n. Show that M is at faithfully flat over R if and only if $\mathfrak{m}M \neq M$ and every system of parameters of R is a regular sequence on M. Note that we are not assuming that M is finitely generated. [One approach to proving “if” is to use reverse induction on i (one knows the result for $i \gg 0$) and Noetherian induction on N to prove that $\mathrm{Tor}^R_i(M, N) = 0$ for all $i \geq 1$ and all Noetherian modules N. Note that every R/P embeds in $R/(f_1,\, \ldots\, , f_h)$ where $f_1,\, \ldots\, , f_h$ is a maximal regular sequence on R in P.]

EXTRA CREDIT 7. Let notation be as in Problem **3.** Show that $G_0(R)$ is generated by $[R]$ and $[R/P]$. Show moreover, that $G_0(R) \cong \mathbb{Z} \oplus \mathbb{Z}$. Find the divisor class group of R.

EXTRA CREDIT 8. Let $(*)$ $\;0 \to A \to B \to C \to 0$ be a short exact sequence of finitely generated modules over a Noetherian ring R. Show that if $B \cong A \oplus C$, possibly in some other way, then the sequence $(*)$ must be split. [Suggestion: First do the case where the three modules have finite length.]