Math 615, Winter 2020 **Problem Set #4 Solutions**

1. Since $0 \to R \xrightarrow{f\cdot} R \to 0$ is a projective resolution of R/fR, we have that $\mathrm{Ext}^1(R/fR,\, M)$ is the cokernel of the map $M \xleftarrow{f\cdot} M$ obtained when we apply $\mathrm{Hom}_R(\_\,,\, M)$ to the resolution, and this is $M/fM \cong (R/fR) \otimes_R M$. In case (a), this is R/fR. In case (b), we get $R/(f, g)R$ in all cases, but this is R/fR if g is a multiple of f and it is R/gR if f is a multiple of g. It does not matter whether g is a nonzerodivisor nor whether f, g is a regular sequence, only that f is not a zerodivisor. In case R is a PID both M and N can be written as a finite direct sum of cyclic modules, where each summand is either R or R/f^hR with the various f occurring are irreducible, and we can do the same with N, where each summand is either R or R/g^kR. We can now distribute Ext^1_R in all possible ways over the pairs of summands. When the input on the left is R, we get 0. Now assume the input on the left is R/f^hR. We get R/f^hR when the input on the right is R. When the input on the left is R/g^k we get 0 if f, g generate distinct prime ideals, since then $(f^h,\, g^k)R$ is the unit ideal. If f and g generate the same prime (we might as well assume they are equal, since each is a unit times the other), we get $R/f^{\min\{h\,,k\}}R$. Alternatively, we have shown that if $M = G \oplus T$, where G is free and T is torsion and so a direct sum of cyclic torsion modules, then $\mathrm{Ext}^1(M, N) \cong T \otimes N$, since $\mathrm{Ext}^1(G, N) = 0$ and the result follows by distributing over the cyclic torsion direct summands of T.

2. The fact that R is regular is not needed, only that it is local. Take a minimal free resolution of M. We can compute the various Ext modules by applying $\mathrm{Hom}_R(\_\,,\, K)$ and taking cohomology. When we map G to K, if $\mathfrak{m}$ is the maximal ideal of R, $\mathfrak{m}P$ maps to 0, and $\mathrm{Hom}_R(G, K) \cong \mathrm{Hom}_K(K \otimes_R G,\, K)$. (One may also think of this a the basic result on base change.) Hence if $G_i \cong R^{b_i}$ is the i th module in the minimal free resolution, $\mathrm{Hom}_R(G_i, K) \to \mathrm{Hom}_R(G_{i+1},\, K)$ is the dual into K of the map $d_i : K \otimes_R G_i \to K \otimes_R G_{i+1}$. All of the d_i are 0 because the resolution is minimal and the matrices have entries in $\mathfrak{m}$. Thus, when we apply $\mathrm{Hom}_R(\_\,,\, K)$ to the minimal free resolution, all of the maps are 0. Hence, $\mathrm{Ext}^i_R(M,\, K)$ may be identified with the dual into K of $K \otimes_R G_i$. The dimension of this K-vector space is the same as the rank of b_i of the free R-module G_i.

3. We have two obvious relations on x, y, namely $(y, -x)$ and $(u, -v)$. Two check that these span all relations, suppose $FX + GY = H(XU - YV)$ in S, so that $X(F - HU) + Y(G + HV) = 0$. This shows that $F - HU$ is a multiple of Y, say LY, and $F = HU + JY$. This shows that if $fx + gy = 0$ then $f = hu + jy$, and so by subtracting $j(y, -x)$ and $h(u, -v)$ we obtain a new relation in which the coefficient of x is 0. Since we are in a domain, the coefficient of y is also 0. Thus, the two obvious relations are the only ones. This means that that we have an exact sequence $R^2 \xrightarrow{A} R^2 \to P \to 0$ where the matrix $A = \begin{pmatrix} y & u \\ -x & -v \end{pmatrix}$ has these relations as columns. Here, the standard generators of R^2 map to x and y, respectively. The map to P is the restriction of the map $R^2 \to R$ with matrix $C = \begin{pmatrix} x & y \end{pmatrix}$, and $CA = 0$. To continue the resolution we need to find the relations on the columns of A. The two columns are proportional over the fraction field of A: the second is $u/y = v/x$ times the first, which means that these are the same as the relations on y and u (or x and v — note that the minus signs have no effect). From the symmetries of the equation $XU - YV$ in the variables it follows that these are generated by $(-v, x)$ and $(-u, y)$. Hence, if we use these as the columns of $B = \begin{pmatrix} -v & -u \\ x & y \end{pmatrix}$ we get the next matrix needed in the free resolution. The columns of this matrix are again proportional: the second is $u/v = y/x$ times the first, and so the relations on these two columns are the same as the relations on x and y. Hence, the next matrix needed is A again. It follows by a completely straightforward induction that the free resolution consists entirely of copies of R^2, with the matrices of the maps alternating between A and B:

$$\cdots \to R^2 \xrightarrow{B} R^2 \xrightarrow{A} R^2 \xrightarrow{B} R^2 \xrightarrow{A} R^2 \xrightarrow{B} R^2 \xrightarrow{A} R^2 \to 0.$$

We have omitted the augmentation P. To compute $\mathrm{Ext}_1^R(P, Q)$ we apply $\mathrm{Hom}_R(\_\,, Q)$. We keep track only of what happens around the spot indexed by the number 1, which corresponds to the occurrence of R^2 above that is second from the right. We have:

$$Q^2 \xleftarrow{B^{\mathrm{tr}}} Q^2 \xleftarrow{A^{\mathrm{tr}}} Q^2$$

We need to calculate the cohomology at the middle spot. The kernel of B^{tr} consists of elements (s, t) where $s, t \in Q$ such that $B^{\mathrm{tr}} \begin{pmatrix} s \\ t \end{pmatrix} = 0$, which means that (s, t) is a relation on the columns of B^{tr}, which are the rows of B. The rows are proportional, so that the relations are the same as those on $-v$ and x. If we allow coefficients in R we know these are spanned by (x, v) and (y, u): these correspond to the columns of A^{tr} with one change of sign. Therefore we need to find all R-linear combinations $f(x, v) + g(y, u)$ of these two vectors that have entries in Q. If we look at this condition in $R/Q \cong K[x, y]$ we see that we need $\overline{f}(x, 0) + \overline{g}(y, 0) = (0, 0)$, where $\overline{f}$, $\overline{g}$ are the images of f, g modulo Q. Hence, mod Q, (f, g) is a multiple of $(y, -x)$. It follows that the relations are spanned by $u(x, v), v(x, v), u(y, u), v(y, u)$ and $y(x, v) - x(y, u) = (0, 0)$, so the last is not needed. This shows that all relations are in the image of the map A^{tr} applied to $Q^2 \cong Q \oplus Q$. Hence, $\mathrm{Ext}^1(P, Q) = 0$.

An approach that does not succeed is to note $\mathrm{Ext}_R^1(P, Q) \cong \mathrm{Ext}^2(R/P, Q)$. Since the depth of Q (which is a maximal Cohen-Macaulay module module) on P is 1, we can conclude $\mathrm{Ext}^1(R/P,\, Q) \neq 0$, but, *a priori*, higher Ext modules may or may not vanish.

Remark. It is actually the case that $P \cong Q$ as R-modules. $P \cong uP = (ux, uy)R = (vy, uy \cong)R = yQ \cong Q$.

4. Let $N = M/(f_1, \ldots, f_{h-1}M$. Then f_h is not a zerodivisor on N, and we need to show that $[N/f_hN] = 0$. But from the short exact sequence $0 \to N \xrightarrow{f_h \cdot} N \to N/f_hN \to 0$ we have that $[N/f_hN] = [N] - [N] = 0$. $\square$

5. Let P_i be the minimal prime generated by the image of x_i. We have a map $G_0(R/P_i) \to G_0(R)$ whose image contains all $[R/Q]$ such that $Q \supseteq P_i$. Since every prime contains a minimal prime, $G_0(R)$ is spanned by the images of the groups $G_0(R/P_i)$. Since R/P_i is a polynomial ring, $G_0(R/P_I)$ is spanned by $[R/P_i]$. It follows that $G_0(R)$ is spanned by the n elements $[R/P_i]$, $1 \le i \le n$. It will suffice to show that there are no relations on these. Suppose $\sum_{i=1}^n h_i[R/P_i] = 0$. Since for every i, we have an additive map $\theta_i : M \mapsto \ell_{R_{P_i}}(M_{P_i})$ which induces a map $G_0(R) \to \mathbb{Z}$ which is 1 on R/P_i and 0 on $[R/P_j]$ if $j \neq i$, it follows that $h_i = 0$, $1 \le i \le n$. $\square$

6. In the presence of the condition $\mathfrak{m}M \neq M$ faithful flatness is equivalent to flatness and improper regular sequences on M are regular sequences. If M is flat, $\_\,_R M$ preserves whether elements are nonzerodivisors and commutes with quotients: hence it preserves (possibly) improper regular sequences. Since every system of parameters for R is regular local (hence, Cohen-Macaulay module), every system of parameters (or part thereof) is a regular sequence, and this remains true on M.

It remains to show that if every system of parameters in R is a regular sequence on M, then M is flat. Let R-modules $A \subseteq B$ be given. Then B is the directed union of its submodules $B_0 \supseteq A$ such that B_0 is generated by A and finitely many more elements. Thus, it suffices to show that injectivity is preserved when B/A is finitely generated, Hence, from the long exact sequence for Tor, it suffices to show that for every finitely generated R-module $N = B/A$, $\mathrm{Tor}_1^R(N,\, M) = 0$. We shall prove that for all finitely generated modules N and $i \ge 1$, we have that $\mathrm{Tor}_i^R(N,\, M) = 0$. Note that we already know this if $N \cong R/(x_1,\, \ldots, x_h)R$, where $x_1,\, \ldots, x_h$ is part of a system of parameters: the Koszul complex gives a free resolution of $N = R/(x_1,\, \ldots, x_h)R$ that case, and the modules $\mathrm{Tor}_i^R(N,\, M)$ are the same as Koszul homology, which vanishes for $i \ge 1$ because $x_1,\, \ldots, x_i$ is a regular sequence on M. Also note that $\mathrm{Tor}_i(N,\, M) = 0$ for $i > \dim(R)$, since R is regular and $\mathrm{pd}_R N \le \dim(R)$. It will suffice to show that if $\mathrm{Tor}_{i+1}(N,\, M) = 0$ for all finitely generated modules N with $i \ge 1$, then $\mathrm{Tor}_i(N,\, M) = 0$ for all finitely generated modules N. We shall use induction on the number of factors in a prime cyclic filtration of N. Note that if the number of factors is $s+1$, we have a short exact sequence $0 \to N_0 \to N \to R/P \to 0$ where N_0 has a prime cyclic filtration with s factors and P is prime. The long exact sequence for Tor implies at once that if $\mathrm{Tor}_i^R(N_0,\, R/P) = 0$ and $\mathrm{Tor}_I^R(R/P,\, M) = 0$, then $\mathrm{Tor}_i^R(N,\, M) = 0$, and the Krull dimensions of N_0 and R/P are at most that of N. We have therefore reduced to the case where $N = R/P$. If $\dim(R/P) = 0$ then $P = \mathfrak{m}$ is generated by a regular sequence, and we are done. Assume the height of P is $h < \dim(R)$ and choose a maximal regular sequence $x_1,\, \ldots, x_h$ in P. Then, since R is Cohen-Macaulay, the height of P is h, and it follows that P is a minimal prime of $(x_1,\, \ldots, x_h)R$ and so an associated prime of $x_1,\, \ldots, x_h)R$. Hence, we have a short exact sequence $0 \to R/P \to R/(x_1,\, \ldots, x_h)R) \to W \to 0$. Then part of the long exact sequence for Tor is $\cdots \to \mathrm{Tor}_{i+1}^R(W,\, M) \to \mathrm{Tor}_i^R(R/P,\, M) \to \mathrm{Tor}_i^R\big(R/(x_1,\, \ldots, x_h)R,\, M\big) \to \cdots$, and since the term on the left is 0 by the induction hypothesis on i and the term on the right is 0 by the fact noted above, the middle term is 0 as well. $\square$