Math 615, Winter 2020 **Problem Set #5**
Due: Tuesday, April 21

1. Let R be a Noetherian domain of prime characteristic $p > 0$, $I,\ J \subseteq R$, and let $f \in R$.

(a) Show that if I is tightly closed, then $I :_R J$ is tightly closed.

(b) Show that if $f \neq 0$ and $g \in (f)^*$, then g/f is integral over R. (You may assume the fact that the integral closure of R is an intersection of Noetherian discrete valuation rings.)

(c) Show that if R is integrally closed, then $(fI)^* = f(I^*)$.

2. [Alternate treatment of *type*.] Let $(R,\, m,\, K)$ be Cohen-Macaulay of dimension n.
(a) Show that given two systems of parameters for R, there is a chain of systems of parameters joining them such that any two consecutive systems in the chain agree except for at most one element. [Suggestion. To start, given systems $x_1,\, \ldots\, , x_n$ and $y_1,\, \ldots\, , y_n$ show there exists z such that $z, x_2,\, \ldots\, ,\, x_n$ and $z, y_2,\, \ldots .\, ,\, y_n$ are both systems of parameters.]

(b) Let $\underline{x} = x_1,\, \ldots\, , x_n$ and $\underline{y} = y_1,\, \ldots\, , y_n$ be two systems of parameters. Let $\mathfrak{A}_{\underline{x}}$ be the annihilator of m in $R/(\underline{x})R$, with similar notation for $\underline{y}$ and other systems of parameters. If $n = 1$, prove that multiplication by y_1 on the numerators induces an injective map $R/x_1R \to R/x_1y_1R$ that carries $\mathfrak{A}_{x_1} \cong \mathfrak{A}_{x_1y_1}$. Hence, by symmetry, $\mathfrak{A}_{x_1} \cong \mathfrak{A}_{y_1}$.

(c) Show that $\mathfrak{A}_{\underline{x}} \cong \mathfrak{A}_{\underline{y}}$ in general. [Suggestion: make use of part (a)]

(d) If $y_i = x_i^{t_i}$ with $t_i \geq 1$ for $1 \leq i \leq n$, show that the map $R/(\underline{x})R \to R/(\underline{y})R$ induced by multiplication by $x_1^{t_1-1} \cdots x_n^{t_n-1}$ on numerators induces an injection that carries $\mathfrak{A}_{\underline{x}} \cong \mathfrak{A}_{\underline{y}}$.
[Thus, $\dim_K(\mathfrak{A}_{\underline{x}})$ is independent of the choice of $\underline{x}$, which justifies calling it the *type* of the Cohen-Macaulay local ring R. Cohen-Macaulay local rings of type 1 are called *Gorenstein.*]

3. Let $(R,\, m,\, K)$ be a Cohen-Macaulay local domain of prime characteristic $p > 0$. Let $x_1,\, \ldots\, , x_n$ be a system of parameters. Suppose that $(x_1,\, \ldots\, , x_n)R$ is tightly closed. Prove that for every system of parameters $y_1,\, \ldots\, , y_n$, the ideal $(y_1,\, \ldots\, , y_n)R$ is tightly closed.

4. Let R be a ring of prime characteristic $p > 0$. Fix $e \geq 1$, and let $q = p^e$. Let $\mathcal{F}^e$ denote the covariant right exact functor from R-modules to R-modules which is base change from $R \xrightarrow{F^e} R$ via the e th iteration of the Frobenius endomorphism. (More generally, if S is an R-algebra, one has the functor base change functor $M \mapsto S \otimes_R M$. Here, $S = R$ but the map $R \to S$ is *not* the identity.) Note that $\mathcal{F}^e(R) = R$, that $\mathcal{F}^e\big(\mathrm{Coker}\,(r_{i,j})\big) \cong \mathrm{Coker}\,(r_{i.j}^q)$, and that $\mathcal{F}^e(R/I) \cong R/I^{[q]}$.
Now also assume that R is regular, which implies that $F^e : R \to R$ is flat (in fact, faithfully flat). Let $M,\, N$ be finitely generated R-modules.

(a) Show that the sets of associated primes of M and $\mathcal{F}^e(M)$, are the same. Hence, if $I \subseteq R$ is an ideal, then $I^{[q]}$ is primary to a prime P if and only if I is primary to P.

(b) Show that $\mathcal{F}^e\big(\mathrm{Ext}^i_R(M, N)\big) \cong \mathrm{Ext}^i_R\big(\mathcal{F}^e(M), \mathcal{F}^e(N)\big)$

5. A ring R of prime characteristic $p > 0$ is called F*-split* if the Frobenius endomorphism $F : R \to R$ is injective and $F(R) = R^p$ is a direct summand of R as $F(R)$-modules.

(a) A domain R is called *seminormal* if whenever f is in the fraction field of R and $f^2,\, f^3 \in R$, then $f \in R$. Prove that an F-split domain is seminormal.

(b) Let K be a field of characteristic $p > 0$ which we assume, for simplicity, is perfect. Prove that the polynomial ring $R = K[x_1,\, \dots,\, x_n]$ is F-split, and that R/I is F-split if I is generated by square-free monomials.

(c) Let R be a polynomial ring as in part (b), let G be a subgroup of the group of permutations of $x_1,\, \dots,\, x_n$, and let G act on R by K-algebra automorphisms that extend its action on the set of variables. Must R^G be F-split? Prove your answer.

6. Let R be a reduced Noetherian ring of characteristic $p > 0$. An element $c \in R$ not in any minimal prime is called a *test element* if for every ideal I and $u \in I^*$, we have that $cu^q \in I^{[q]}$ for all $q = p^e$, $e \in \mathbb{N}$. [By theorems beyond the scope of this course, test elements exist in many cases: for example if R is a localization of a finitely generated algebra over a ring B that is a field or a complete local ring B (in fact, it suffices if B is semilocal and excellent).]

(a) Show that if R is F-split as in Problem **5.** and c^n is a test element for some $c \in R$, then c is a test element.

(b) Show that if $(R, \mathfrak{m}, K)$ is local and $c \in R$ is a test element, then a proper ideal I is tightly closed if and only if I is an intersection of tightly closed $\mathfrak{m}$-primary ideals.

EXTRA CREDIT 9. Fix $n \geq 3$ and integers $a_1, \dots, a_n > 0$. Let K be a field of characteristic $p > 0$ (p will vary) and let $f = x_1^{a_1} + \cdots + x_n^{a_n}$. Let $R = K[x_1,\, \dots,\, x_n]/(f)$, and let $I = (x_1,\, \dots,\, x_{n-1})R$. Let $\alpha = \frac{1}{a_1} + \cdots + \frac{1}{a_n}$. Show that if $\alpha \leq 1$, then $x_n^{a_n - 1}$ is in the tight closure I^* of I. Show that if $\alpha > 1$ then for all sufficiently large prime integers p, the element $x_n^{a_n-1} \notin I^*$. [For the last statement you may use the following fact: let F be a polynomial that is irreducible even over the algebraic closure of K. Then all of the images of the partial derivatives $\partial F/\partial x_j$ in R/FR, if nonzero, are test elements for R/FR.]

EXTRA CREDIT 10. Given an example of a normal domain R finitely generated over a field K of prime characteristic $p > 0$ and two ideals $I,\, J \subseteq R$ such that $I^{[p]} :_R J^{[p]} \neq (I :_R J)^{[p]}$. (By a class result, this does not happen if R is regular.)