Math 615, Winter 2020 **Problem Set #5: Solutions**

1. (a) If $u \in (I :_R J)^*$, there exists $c \neq 0$ in R such that for all $q \gg 0$, $cu^q \in (I :_R J)^{[q]} \subseteq I^{[q]} :_R J^{[q]}$. Hence, if $j \in J$, for all $q \gg 0$, $j^q(cu^q) = c(ju)^q \in I^{[q]}$, so that $ju \in I^* = I$. Since this holds for all $j \in J$, $u \in I :_R J$. $\square$

(b) If this is false, we can find a Noetherian discrete valuation ring $V \supseteq R$ such that $f/g \notin V$. Since, in V, $f \in (gV)^*$, it will suffice to show that gV is tightly closed. If $g = 0$ this is clear. Otherwise, choose $cd \neq 0$ such that $cf^q \in g^qV$ for all $q \gg 0$. Then $\mathrm{ord}\,(c) + q\,\mathrm{ord}\,(f) \geq q\,\mathrm{ord}\,(g)$ and $\mathrm{ord}\,(f) \geq \mathrm{ord}\,(f) - \mathrm{ord}\,(c)/q$ for all $q \gg 0$. It follows that $\mathrm{ord}\,(f) \geq \mathrm{ord}\,(g)$, as required. $\square$

(c) If $f = 0$, this is clear. Assume $f \neq 0$. $(fI)^* \subseteq (fR)^* = fR$ by (b). But if $r \in R$, $fr \in (fI)^*$ iff for some nonzero c and all $q \gg 0$, $c(fr)^q \in (fI)^{[q]} = f^qI^{[q]}$, i.e., $cr^q \in I^{[q]}$. Thus, $fr \in (fI)^*$ iff $r \in I^*$, and the result follows. $\square$

2. (a) We use induction on n. The case $n = 1$ is obvious. Following the suggestion, first note that all minimal primes of $(x_2, \dots, x_n)$ and of $(y_2, \dots, y_n)$ have height $n - 1$ (the quotient of a local ring R by the ideal generated by i elements of a system of parameters has dimension $n - i$), and so the maximal ideal is not covered by their union. Choose z in the maximal ideal not in any of these primes. Then $(z, x_2, \dots, x_n)$ and $(z, y_2, \dots, y_n)$ must both be m-primary. Clearly, it suffices to construct such a chain of systems of parameters between $(z, x_2, \dots, x_n)$ and $(z, y_2, \dots, y_n)$. But this is equivalent to getting such a chain between the images of $x_2, \dots, x_n$ and $y_2, \dots, y_n$ working in R/zR, and we may apply the induction hypothesis.

(b) We omit the subscript $_1$. Multiplcation by y carries R isomorphically onto yR (since y is a nonzerodivisor) and $xR \subseteq R$ isomorphically onto xyR. Hence, the map induced by multiplication by y carries R/xR isomorphically on $yR/xyR \subseteq R/xyR$. In particular, the specified map is an injection, and must take $\mathfrak{A}_x = \mathrm{Ann}_{R/xR}m$ into $\mathfrak{A}_{xy} = \mathrm{Ann}_{R/xyR}m$. Suppose u in R is such that $mu \subseteq xyR$, and $\overline{u}$ is its image in R/xyR. Then $xu \in xyR$, and since x is a nonzerodivisor, $u \in yR$. Hence, $\overline{u} \in \mathrm{Ann}yR/xyRm$, and the isomorphism $R/xR \cong yR/xyR$ shows that $\overline{u}$ must be the image of an element of $\mathfrak{A}_x$. $\square$

(c) Using the parenthetical comment, we may reduce the problem at once to the case where the two systems are the same except for one element. Since the order of the parameters is not relevant, we may assume that the systems are $x_1, \dots, x_{n-1}$, x and $x_1, \dots, x_{n-1}y$. The issue is unaffected if we replace R by $\overline{R} = R/(x_1, \dots, x_{n-1})R$ and x, y their images in $\overline{R}$, each of which is a parameter for the one-dimensional Cohen-Macaulay ring $\overline{R}$. The result is then immediate from part (a). $\square$

(d) It suffices to prove the result when all the t_j except t_i are equal to one: one may then apply that result n times, one time for each x_j, to obtain the desired fact. We may proceed by placing x_j last in the sequence, and working modulo the other elements of the sequence, as in part (b), to reduce to the case where $n = 1$. The result then follows from taking x to be the image of x_j and y to be $x_j^{t_j-1}$. $\square$

3. As in Problem 2., it suffices to consider the case where the two systems of parameters are the same except for one element, and one can compare each of $I = x_1, \dots, x_{n-1}$, x and

$x_1, \ldots, x_{n-1}, y$ with $J = x_1, \ldots, x_{n-1}, xy$. If an m-primary ideal I is not tightly closed, let u be an element of the tight closure not in I. Then either $mu \subseteq I$, are we can choose $z_1 \in m$ such that $z_1 u \notin I$, while it is still true that $z_1 u \in I^*$. Repeating this process finitely many times, we can find an element $u \in I^* - I$ such that $mu \subseteq I$: the process cannot continue more than N steps, where $m^N \subseteq I$. Thus, I fails to be tightly closed if and only if there is an element of $I :_R m$ that is in $I^* - I$. By the result of Problem 2., we have a bijection between $(I : m)/I$ and $(J : m)/I$ induced by multiplication by y. Thus, it suffices to show for $u \in I : m$ that $u \in I^*$ if and only if $yu \in J^*$. But $c(yu)^q \in (x_1^q, \ldots, x_{n-1}^q, x^q y^q)$ iff $cu^q \in (x_1^q, \ldots, x_{n-1}^q, x^q y^q) : y^q = (x_1^q, \ldots, x_{n-1}^q, x^q)$ (one may check this working mod $\mathfrak{A} = (x_1^q, \ldots, x_{n-1}^q)$, where it follows from the fact that x and y are both nonzero divisors mod $\mathfrak{A}$), and the result is now immediate.

4. For part (a), suppose that $R/P \hookrightarrow M$. Then $R/P^{[p^e]} \cong \mathcal{F}^e(R/P) \hookrightarrow \mathcal{F}^e(M)$, and since the radical of $P^{[p^e]}$ is P, P is a minimal and hence associated prime of $R/P^{[p^e]}$, and we have $R/P \hookrightarrow R/I^{[p^e]} \hookrightarrow \mathcal{F}^e(M)$, as required. Now assume that P is not an associated prime of M. Localization at P commutes with $\mathcal{F}^e$, and so may assume that (R, P) is local with maximal ideal P, that P is not an associated prime of M, and we want to show that P is not an associated prime of $\mathcal{F}^e(M)$. This follows because we can choose $x \in P$ not a zerodivisor on M, and then x^{p^e} (equivalently, x) is not a zerodivisor on $\mathcal{F}^e(M)$. $\square$

(b) Let S be R thought of as an algebra via F^e over R. By flat base change to S for $\mathrm{Ext}^i R(M, N)$ (which holds whenever M is finitely generated over a Noetherian ring R and S is an R-flat algebra), we have $S \otimes_R \mathrm{Ext}^i R(M, N) \cong \mathrm{Ext}^i_S(S \otimes_R M, S \otimes_R N)$, which yields the stated result at once. $\square$

5. Note that when $R \to S$ is a ring homomorphism that splits over R, then $IS \cap R = R$: see part (a) of the Lemma on p. 123 of the Lecture Notes from March 11. Also note that when $F : R \to R$ is split (which implies that R is reduced), then each map in the composite $R \xrightarrow{F} R \xrightarrow{F} R \xrightarrow{F} \cdots \xrightarrow{F} R$ is also split, and so $R \xrightarrow{F^e} R$ is split as well. It follows that if S is R viewed as an R-algebra via F^e and S is F-split over R then $IS \cap R = I$, which means that if the image of r in S, which is r^{p^e}, is in $IS = I^{[p^e]}$, then $r \in I$. That is, if R is F-split, $R \in R$, and $I \subseteq R$ is an ideal, then $r^{p^e} \in I^{[p^e]}$ implies that $r \in I$.

(a) Suppose that R is an F-split domain and $r^3, r^2 \in R$. This implies that $r^n \in R$ for all $n \geq 2$. Now $(r^3)^p \in (r^2 R)^{[p]} = r^{2p} R$, since $r^{3p+=r^{[2p}} r^p$ and $r^p \in R$. From the first paragraph, this implies that $r^3 \in r^2 R$, and so $r^3/r^2 = r \in R$. $\square$

(b) The image of the injection $F : R \to R$ is $K[x_1^p, \ldots, x_n^p] \subseteq K[x_1, \ldots, x_n]$. Let W be the K-span in R of the set of monomials $\mathcal{M}$ such that the degree in some variable is not divisible by p. Then $R = F(R) \oplus W$, as vector spaces over K, and since W is an $F(R)$-module (the product of a monomial in which every exponent is divisible by p and a monomial in $\mathcal{M}$ is in $\mathcal{M}$, and the result follows from the distributive law). Hence, $R \xrightarrow{F} R$ is F-split. Call the splitting $\theta : R \to F(R)$ (but note that to give the splitting of the map $R \xrightarrow{F} R$, one must compose θ with $F^{-1} : F(R) \cong R$).

Let I be an ideal generated by square-free monomials. Then θ maps I into I, since I is the K-span of monomials each which maps to itself or to 0 under θ. Hence, θ induces a map $R/I \to F(R/I)$ that fixes every element of $F(R/I)$. $\square$

(c) Consider any monomial μ occurring in an element of $f \in R^G$. Then every monomial in the orbit of μ must also occur in f with the same coefficient as μ. It follows that if Σ denotes the set of orbit sums of monomials under the action of G, then Σ is a K-basis for R^G. Consider the splitting $\theta : R \to F(R)$ introduced in part (b) of this problem. Then θ maps Σ to itself, fixing the orbit sums of monomials that are pth powers, and killing the orbit sums of the other elements of Σ. It follows that restricting the domain and range of Σ gives a splitting of $F(R^G) \to R^G$, so that R^G *is* F-split.

6. (a) The elements c and c^n belong to the same primes, and so one is in R° if and only if the other is. If we choose $q_0 \geq n$, then c^{q_0} is a multiple of c^n and is also a test element. Let I be an ideal of R and $u \in I^*$. Then for all q, $c^{q_0}u^q \in I^{[q]}$. This also holds if we replace q by qq_0. It follows that for all q, $cq_0u^{qq_0} = (cu^q)^{q_0} \in I^{[qq_0]} = (I^{[q]})^{[q_0]}$, and now we may apply the fact that $R \xrightarrow{F^{q_0}} R$ is split and the remark in the first paragraph of the solution of Problem **5.** to conclude that for all q, $cu^q \in I^{[q]}$, as required. $\square$

(b). Call the local ring $(R, \mathfrak{m})$. It is immediate from the definition of tight closure that an intersection of tightly closed ideals is tightly closed, so that "if" is obvious. To prove "only if" it will suffice to show that if $I \subseteq \mathfrak{m}$ is tightly closed, then $I = \bigcap_{n=1}^{\infty}(I + \mathfrak{m}^n)^*$. One inclusion, namely $\subseteq$, is clear. To prove $\supseteq$, we need to show that if $f \in (I + \mathfrak{m}^n)^*$ for all n then $f \in I^*$. Let c be a test element. Then for all n and for all q, $cf^q \in (I + \mathfrak{m}^n)^{[q]} \subseteq I^{[q]} + (\mathfrak{m}^n)^{[q]} \subseteq I^{[q]} + \mathfrak{m}^n$. For every fixed q, this holds for all $n \geq 1$, and since $I^{[q]}$ is $\mathfrak{m}$-adically closed, this implies that for all q, $cf^q \in I^{[q]}$, and so $f \in I^*$, as required. $\square$