J. ALGEBRAIC GEOMETRY
3 (1994) 599–670

TIGHT CLOSURE OF PARAMETER IDEALS AND SPLITTING IN MODULE-FINITE EXTENSIONS

MELVIN HOCHSTER AND CRAIG HUNEKE

Table of Contents

1. Introduction

Throughout this paper, unless otherwise specified, rings are assumed to be commutative, associative, with identity, and modules are assumed to be unital. For the most part we shall work with Noetherian rings of positive prime characteristic p. Suppose that R is such a ring. We recall that an element $x \in R$ is said to be the *tight closure* of an ideal I of R if there exists an element c in R but not in any minimal prime ideal of R such that for all sufficiently large integers $q = p^e$, where $e \in \mathbb{N}$, the nonnegative integers, we have $cx^q \in I^{[q]}$ (the ideal generated by the qth powers of the elements of I). R is called *weakly* F*-regular* if every ideal is tightly closed, and F*-regular* if it and all of its localizations are weakly F-regular.

We use the theory of tight closure to develop the notion of when an extension of finitely generated modules over R is a *phantom extension.* Using these ideas, we prove a number of splitting theorems. For example, we are able to show that if a ring is weakly F-regular then it is a direct summand, as a module over itself, of every module-finite extension ring.

Received April 15, 1992 and, in revised form, June 1, 1993. The authors were supported in part by grants from the National Science Foundation.

This result generalizes the fact that a regular ring of characteristic p is a direct summand of every module-finite extension ring. We also show that a locally excellent Gorenstein ring is weakly F-regular if and only if it is a direct summand of every module-finite extension ring. The results developed toward proving this can be used to study the problem of when an $\mathbb{N}$-graded Noetherian ring is weakly F-regular.

We also use the notion of phantom extension to give a proof of the existence of big Cohen-Macaulay modules. This proof is closely related to the original proof, but we feel that it is worth giving the argument as formulated here because doing so shows that a "sufficiently good" notion of tight closure in mixed characteristic would also yield a proof of the existence of big Cohen-Macaulay modules in mixed characteristic.

The authors introduced, in [HH4], the notion of the tight closure of an ideal or submodule, both for Noetherian rings of positive prime characteristic p and for finitely generated algebras over a field of characteristic 0. Accounts that are at least partly expository are given in [HH1–3], [Hu2], and [Ho10]. The theory is developed further in [HH6], [HH8], [HH9], [HH10], and [HH11], as well as in [Ab1–3], [AHH], [Gla], and [Ho9] (where a theory of so-called *solid closure* is developed, defined in all characteristics, even mixed characteristic, and coinciding with tight closure in good cases in characteristic p), [Sm1–3], [Vel], and [Wil]. In particular, [HH8] contains a detailed study of the notion of *phantom homology* (homology is *phantom* when the cycles are in the tight closure of the boundaries in the module of chains), which leads to the theory of modules of *finite phantom projective dimension* developed in [Ab1–2]. Some results on the commutativity of localization with tight closure for modules $N \subseteq M$ when M/N has finite phantom projective dimension are obtained in [AHH]. We shall make strong use here of the results of [HH9], which deals both with the extension of test elements (see (3.5), (3.6), and (3.7)) and with the behavior of tight closure under smooth base change.

Tight closure techniques have produced new proofs that rings of invariants of linearly reductive groups acting on regular rings are Cohen-Macaulay (cf. [HR1–2], [Ke], [Bou], [HoE], and [HH4]), of the Briançon-Skoda theorem on integral closures of ideals in regular rings in a greatly strengthened form in the equicharacteristic case (cf. [BRS; HH4, §5; LS; LT; Sk] for background), and of various local homological theorems (known in the equicharacteristic case and conjectured in mixed characteristic: cf. [PS1–2], [Ro1–5], [Ho1–3], [Ho5–7], [EvG1–3], and [Du] for further information) in much improved forms: see [HH4, §10; HH8, §§4–6]. In every instance, the new methods have also yielded new insight.

Tight closure enables one to control the results of performing operations on ideals generated by monomials in a system of parameters, even repeated operations, when the parameters are not necessarily a regular sequence. One can often show that the operations produce an ideal that is in the tight closure of the "formal" answer one would get if the parameters did form an R-sequence. See §7 of [HH4] for details.

In [Hu3] tight closure is a crucial ingredient in establishing unexpectedly strong uniform Artin-Rees theorems. It is also worth mentioning that the study of tight closure led to the discovery of the Cohen-Macaulay property for absolute integral closures R^+ of excellent local domains R in characteristic p (see [HH7] and the final paragraph of this introduction).

The paper is organized as follows: §2 contains some basic notation and conventions, particularly concerning rings of characteristic p. In §3 we discuss briefly some definitions and results on tight closure mainly from [HH3–4], [HH9] that are needed throughout. The graded case is dealt with in §4.

In §5 we introduce the notion of a phantom extension and use it both to prove the existence of big Cohen-Macaulay modules in characteristic p and to establish splitting theorems for module-finite extensions of weakly F-regular rings. We also study a closely related question: when is it true that for a ring extension $R \to S$ that $IS \cap R \subseteq I^*$ for every ideal I of R, or, even better, that $(IS)^* \cap R \subseteq I^*$ for every ideal I of R. There is also a corresponding question for modules. It turns out that there are very good results for module-finite extensions, and, more generally, for extensions that "preserve" height sufficiently well. See Theorem (5.22), Corollary (5.23), Theorem (5.31), and Theorem (5.32).

In §6 we discuss some results on forcing the elements of the tight closure I^* of an ideal I of a ring R into IS for a suitable integral extension ring $S \supseteq R$, and we use these results to prove that if R is locally excellent and Gorenstein, then R is F-regular if and only if R is a direct summand, as a module over itself, of every module-finite extension. (This problem was also studied in [Ho2] and [Ma].)

In §7 some of the results of §6 are used to develop criteria for F-regularity, which are then applied to the study of a number of examples. Some of the results obtained overlap results of [FeW].

In §8 we study integral closures of ideals by tight closure techniques. We develop a new generalization of the Briançon-Skoda theorem for Cohen-Macaulay rings with isolated singularities and related results. We investigate a theorem of Itoh and Huneke using tight closure techniques, and we prove a tight closure analogue of this result. We also use a result of

Lipman and Sathaye to give a useful method for finding test elements.

This paper deals exclusively with tight closure theory in characteristic p. The characteristic 0 theory will be developed in [HH10].

Finally, we mention one more result that gives a very useful perspective on tight closure. The authors have recently proved (see [HH7]; [HH5] and [Hu4] are expositions) that if R is an excellent local domain of characteristic p, then the integral closure R^+ of R in an algebraic closure of the fraction field of R (which is called the *absolute integral closure* of R) is a big Cohen-Macaulay algebra for R. Our results here imply that if I is an ideal of R then $IR^+ \cap R \subseteq I^*$. So far as the authors know, it is possible that $I^* = IR^+ \cap R$ in very great generality. This is proved for n generator parameter ideals when R is locally excellent for $n \leq 3$ in §6 of [HH11] and for all n in [Sm1–2].

It is worth mentioning that one of the crucial techniques used in the proof that R^+ is a big Cohen-Macaulay algebra (which involves giving a criterion for when an ideal $J \supseteq I$ can be forced into the expansion of I in a suitable integral extension ring: see Theorem (6.3) and Theorem (6.6)) was, in fact, first developed for application in §6 of this paper.

There is a fascinating interaction between the study of the properties of tight closure and the study of the behavior of the rings R^+: we expect this interaction to be a persistent phenomenon.

2. Notation and conventions

(2.1) Notation. In any commutative ring R, R° denotes the complement of the union of the set of minimal primes. Thus, if R is a domain, $R^\circ = R - \{0\}$.

(2.2) Convention. By a *local* ring (R, m, K) we mean a Noetherian ring R with a unique maximal ideal m such that $K = R/m$ is the residue field.

(2.3) Definition. A *parameter* in a Noetherian ring R is an element of R°. A sequence of elements $x_1, \cdots, x_n$ is called a *sequence of parameters* if their images form part of a system of parameters in every local ring R_P of R at a prime ideal P containing the ideal $I = (x_1, \cdots, x_n)R$. In this case we refer to I as an *ideal generated by parameters* or as a *parameter ideal*.

(2.4) Discussion. Throughout, p always denotes a positive prime integer.

We make the following conventions for discussing rings of positive prime characteristic p. We shall use e to denote a variable element of the set $\mathbb{N}$ of nonnegative integers and q for a variable element of the set $\{p^e : e \in \mathbb{N}\}$. Thus, "for all e" is synonymous with "for all $e \in \mathbb{N}$" while "for some q" is synonymous with "for some q of the form p^e with $e \in \mathbb{N}$".

Throughout the rest of this section all rings are assumed to be of characteristic p.

(2.5) Notation. We denote by F or F_R the Frobenius endomorphism of a ring R of characteristic p, and we denote by F^e the eth iteration of F, so that $F^e(r) = r^q$ where $q = p^e$. If R is a reduced ring of characteristic p we write $R^{1/q}$ for the ring obtained by adjoining qth roots of all elements of R. The inclusion map $R \subseteq R^{1/q}$ is isomorphic with $F^e: R \to R$.

(2.6) Definition. If $I \subseteq R$ and $q = p^e$, then $I^{[q]}$ denotes the ideal $(i^q : i \in I)R$, which is also the expansion $F(I)R$ of I under the Frobenius map $F: R \to R$. Note that if T denotes a set of generators for I, then $\{t^q : t \in T\}$ generates $I^{[q]}$.

(2.7) Discussion and definition. We shall make considerable use of the Peskine-Szpiro functors $\mathbf{F}^e_R = \mathbf{F}^e$, where R has characteristic p and $e \in \mathbb{N}$. For any R-algebra S, $S\otimes_R\_$ gives a functor from R-modules to S-modules which takes finitely generated R-modules to finitely generated S-modules. When S is R itself viewed as an R-algebra via F^e, the eth iteration of the Frobenius endomorphism, we refer to this functor as the *Peskine-Szpiro* or *Frobenius functor*: see [PS1]. Thus, $\mathbf{F}^e$ is a covariant functor from R-modules to R-modules that preserves finite generation. Note that $\mathbf{F}^e(R) \cong R$, that $\mathbf{F}^e(R^n) \cong R^n$ (not canonically), and that if we apply $\mathbf{F}^e$ to a map $R^n \to R^m$ with matrix (a_{ij}) we obtain a map $R^n \to R^m$ with matrix (a_{ij}^q), where $q = p^e$. The R-module structure on $\mathbf{F}^e(M)$ is such that $r'(r\otimes m) = (r'r)\otimes m$. On the other hand, $r'\otimes(rm) = (r'r^q)\otimes m$. Observe also that $\mathbf{F}^e(R/I) \cong R/I^{[q]}$, where $q = p^e$.

(2.8) Discussion and notation. There is a canonical map $M \to \mathbf{F}^e(M)$ that sends m to $1 \otimes m$. If $q = p^e$ and $w \in M$, we write w^q for the image of w in $\mathbf{F}^e(M)$. With this notation, $(x+y)^q = x^q + y^q$ and $(rx)^q = r^q x^q$. Moreover, $\mathbf{F}^e(R^n)$ may be identified with R^n in such a way that if $w = (a_1, \cdots, a_n)$ then $w^q = (a_1^q, \cdots, a_n^q)$. Thus, this notation for modules is consistent with our notation for the Frobenius endomorphism of R.

If $N \subseteq M$, we shall write $N^{[q]}$ (or, more precisely, $N^{[q]}_M$) for $\mathrm{Im}(\mathbf{F}^e(N) \to \mathbf{F}^e(M))$ $(= \mathrm{Ker}(\mathbf{F}^e(M) \to \mathbf{F}^e(M/N))$ by the right exactness of $\otimes)$. We may also characterize $N^{[q]}$ as the R-span in M of the elements w^q for $w \in N$. Of course, $N^{[q]}$ depends heavily on what M

is (or, more precisely, on what $N \to M$ is). Note that $M_M^{[q]} = \mathbf{F}^e(M)$, and we sometimes write $M^{[q]}$ for $\mathbf{F}^e(M)$. The context should make clear what is meant.

Note that if $M = R$ and $N = I \subseteq R$, then $I_R^{[q]}$ in this notation is the same as what was earlier described as $I^{[q]}$.

3. Tight closure and test elements

This section gives some basic definitions and results, primarily from [HH4] and [HH9], that are needed throughout. All rings are assumed to be Noetherian of characteristic p.

(3.1) Definition. Let $N \subseteq M$ be finitely generated R-modules. We say that $w \in M$ is in the *tight closure* N^* (or, more precisely, N_M^*) of N in M if there exists an element $c \in R^\circ$ (see (2.1a)) such that for all sufficiently large $q = p^e$, $cw^q \in N_M^{[q]}$. In particular, when $N = I \subseteq R$, $w \in I^*$ means that $cw^q \in I^{[q]}$ for all sufficiently large q.

(3.2) Discussion. It is easy to check that the image of N_M^*/N in M/N is the same as the tight closure of the submodule 0 in M/N. This enables one to reduce many problems to the study of the tight closure of the zero submodule. On the other hand, one can map a finitely generated free module G onto M, letting H be the inverse image of N in G, and studying H^* in G: $H^*/H \cong N^*/N$ under the obvious identification $G/H \cong M/N$.

Theorem 3.3 contains some basic information about tight closure that is used repeatedly, often tacitly, throughout the paper.

(3.3) Theorem. *Let R be a Noetherian ring of characteristic p and let $N \subseteq M$ be R-modules.*

(a) *N^* is a submodule of M containing N and $(N^*)^* = N^*$. If $N_1 \subseteq N_2 \subseteq M$, then $N_1^* \subseteq N_2^*$.*

(b) (*Irrelevance of nilpotents*) *If J is the nilradical of R, then $JM \subseteq N^*$. Moreover, if N' denotes the inverse image $(N + JM)/JM$ of N in M/JM, then N^* is the inverse image in M of the tight closure N'^* of N' in M/JM, where N'^* may be computed over either R or R_{red}.*

(c) *If R is reduced or if* $\mathrm{Ann}_R(M/N)$ *has positive height, then $x \in M$ is in N^* if and only if there exists $c \in R^\circ$ such that $cx^q \in N_M^{[q]}$ for all $q = p^e$.*

(d) *An element x of M is in the tight closure of N if and only if, for every minimal prime P of R, the image of x in M/PM is in the tight closure, over R/P, of* $\mathrm{Im}(N/PN \to M/PM)$.

Proof. For parts (a), (b), (c) see [HH4, Proposition 8.5(a), (b), (c), (e), and (j)]. Part (d) is proved for the case of ideals in Proposition (6.25a) of [HH4]. In studying the module case one may assume, as usual, that M is free, and the argument is then the same as for the case of ideals. □

We recall the notions of test element and weak test element.

(3.4) Definition. Let R be a Noetherian ring of characteristic p and let $q' = p^{e'}$ for some integer $e' \in \mathbb{N}$. We say that $c \in R^\circ$ is a *q'-weak test element* if for every finitely generated R-module M and every submodule $N \subseteq M$, an element $x \in M$ is in N_M^* if and only if $cx^q \in N_M^{[q]}$ for all $q \geq q'$.

We say that c is a *locally* (respectively, *completely*) *stable q'-weak* test element if its image in (respectively, in the completion of) every local ring of R is a q'-weak test element.

If c is a q'-weak test element for $q' = 1$ then c is called a *test element.* Corresponding terminology is used in the locally stable and completely stable cases.

(3.5) Remark. By Proposition 8.13(c) of [HH4], if R is a local ring with an element c such that c is a q'-weak test element for $\widehat{R}$, then c is a q'-weak test element for R. This implies that a completely stable q'-weak test element for R is a locally stable q'-weak test element for R.

The next result gives several circumstances in which various kinds of test elements are known to exist.

(3.6) Theorem (existence of test elements). *Let R be a Noetherian ring of characteristic p.*

(a) *Let R be module-finite A-torsion-free and generically smooth over a regular domain A. Then every element $d \in A^\circ$ such that $A_d \to R_d$ is smooth has a power that is a completely stable test element.*

(b) *Let R be such that $R^p \to R$ is module-finite or let R be an algebra essentially of finite type over an excellent local ring. So long as $R \neq 0$ there always exist elements $c \in R^\circ$ such that $(R_{\mathrm{red}})_c$ is regular, and every such element has a power that is a completely stable weak test element. If R is reduced, then every such element c has a power that is a completely stable test element.*

Proof. Part (a) is contained in Theorem 6.13 of [HH4] as generalized in the discussion immediately following Theorem 8.14 of [HH4]. Part (b) for the case where $R^p \to R$ is module-finite follows from Theorem 3.4 of [HH3] as generalized in Theorem 5.10 of [HH9], while part (b) for the case where R is essentially of finite type over an excellent local ring is a consequence of Theorem (6.1b) of [HH9]. □

In §6 we shall need the following result, which is a rephrasing of Corollary 7.34 of [HH9].

(3.7) Lemma. *Let (R, m, K) be a reduced local ring such that, for every $x \in m$, R_x is Gorenstein and F-regular. Suppose that the map $R \to \widehat{R}$ has regular fibers, which holds, in particular, if R is excellent. Then R possesses an m-primary ideal q such that the elements of $q \cap R^\circ$ are test elements. In particular, q is generated by test elements.* □

We also recall the following basic facts about weak F-regularity and F-regularity from [HH4] and [HH9].

(3.8) Proposition. *Let R be a Noetherian, of characteristic p.*

(a) *If R is weakly F-regular then every submodule of every module is tightly closed (the definition only imposes this condition on ideals).*

(b) *R is weakly F-regular if and only if its localization at every maximal ideal is weakly F-regular.*

(c) *If R is regular then it is F-regular. If R is weakly F-regular then R is normal.*

(d) *If R is weakly F-regular and R is either a locally excellent ring or a homomorphic image of a Cohen-Macaulay ring, then R is Cohen-Macaulay.*

(e) *If R is a Gorenstein ring and R is weakly F-regular then it is F-regular.*

(f) *If R is a Gorenstein local ring then in order that R be F-regular it is necessary and sufficient that the ideal generated by a single system of parameters be tightly closed in R.*

Proof. (a), (b), and (c) are Proposition 8.7, Corollary 4.15, and Theorem 4.6 together with Corollary 5.11 of [HH4]. (d), (e), and (f) are Proposition 6.27(b) together with Theorem 3.4(c), and Corollary 4.7(a), (b) of [HH9]. □

4. Tight closure in graded and multigraded rings

Our objective in this section is to study tight closure and F-regularity for multigraded Noetherian rings $R = \bigoplus_{\eta \in H} R_\eta$ (where $H = \mathbb{Z}^h$) and pairs of multigraded modules $N = \bigoplus_{\eta \in H} N_\eta \subseteq M = \bigoplus_{\eta \in H} M_\eta$. The multigrading hypothesis tells us that $R_\eta M_\tau \subseteq M_{\eta+\tau}$ for all $\eta, \tau \in H$. In the most important case, $h = 1$ and $R_\eta = 0$ if $\eta < 0$: this is what we mean by the *graded* case. Note that in the graded case, finitely generated modules may have nonzero negative pieces, but only finitely many.

(4.1) Discussion. (a) If $\alpha = (\alpha_1, \cdots, \alpha_h)$ where the α_j are units of R_0 then α induces a multigraded (this always means degree-preserving rather than degree-shifting) automorphism θ_α that sends each form $r \in R_\eta$ to $\alpha^\eta r$, where if $\eta = (\eta_1, \cdots, \eta_h)$ then $\alpha^\eta = \prod_{j=1}^h \alpha_j^{\eta_j}$. Each θ_α induces a K-linear map (which we also denote by θ_α) from every graded R-module M to itself that sends $m_\eta \in M_\eta$ to $\alpha^\eta m_\eta$. Every multigraded submodule of M is stabilized by all the θ_α. If K is infinite then the converse is true: a submodule N of M that is stable under all the θ_α is multigraded. It is easy to show this by induction: one reduces to the case where $h = 1$. Then, if $m = \sum_{i=a}^{a+d} m_i$ is in N so is $\sum_{i=a}^{a+d} \alpha^i m_i$ for every $\alpha = \alpha_1$, and hence so is $\sum_{i=0}^{d} \alpha^i m_{i+a}$. By choosing $d+1$ distinct values $\alpha(j)$ for α and using the invertibility of the Vandermonde matrix $(\alpha(j)^i)$, we see that every m_i is in N. All we are using about the α_j is that they are units such that the difference of any two distinct α_j is a unit. Notice that for any given element $m \in N$ we only need to use a specific finite number of α_j to prove that all the multigraded pieces of m are in N. Cf. the discussion in (7.30) of [HH9].

This idea will go a long way towards proving that grading is compatible with tight closure operations. When R_0 contains only a finite field we can nonetheless remedy the situation by using the fact that finite separable field extensions are "innocuous" from the point of view of tight closure, and we can make arbitrarily large such extensions.

(b) Let I be a homogeneous ideal in an $\mathbb{N}$-graded ring such that all of its homogeneous elements have positive degree. If the set $\mathcal{F}$ of forms of I is contained in a finite union of homogeneous prime ideals, then I itself must be contained in one of them. (If $\mathcal{F} \subseteq \bigcup_i P_i$ irredundantly, for all i pick $f_i \in (\mathcal{F} \cap P_i) - \bigcup_{j \neq i} P_j$. Taking powers, we may assume $\deg f_i = \deg f_j$ for all i, j. If $\mathcal{A} = \{i : f_1 + f_2 \notin P_i\}$ has r elements then $h = (f_1 + f_2)^r + \prod_{i \in \mathcal{A}} f_i \in \mathcal{F} - \bigcup_i P_i$, a contradiction. $\square$)

(c) We recall from the discussion preceding the statement of Theorem 7.29 of [HH9] that if R is a Noetherian ring of characteristic p, we let $\tau_{q'}(R) = \{c \in R\colon$ if $x \in 0^*$ in some finitely generated R-module M then $cx^q = 0$ in $F^e(M)$ for all $q \geq q'\}$, and we let $\hat{\tau}_{q'}(R)$ denote the intersection of the contractions to R of the ideals $\tau_{q'}(B)$ for all R-algebras B that are completions of local rings of R. R has a q'-weak test element (respectively, a completely stable q'-weak test element) if and only if at least one element of R° is in $\tau_{q'}(R)$ (respectively, $\hat{\tau}_{q'}(R)$), in which case the q'-weak test elements of R (respectively, the completely stable q'-weak test elements of R) are precisely the elements of $\tau_{q'}(R) \cap R^\circ$ (respectively, of $\hat{\tau}_{q'}(R) \cap R^\circ$). When $q' = 1$, $\tau_{q'}(R)$ becomes $\tau(R)$, the ideal of test elements introduced in §8 of [HH4].

(4.2) Theorem. *Let R be a multigraded Noetherian ring of characteristic p.*

(a) *For all q', $\tau_{q'}(R)$ and $\hat{\tau}_{q'}(R)$ are multihomogeneous.*

(b) *If $N \subseteq M$ are finitely generated multigraded modules such that the inclusion preserves degree then N_M^* is multigraded.*

(c) *Suppose that $\{P \in \operatorname{Spec} R : R_P$ is weakly F-regular$\}$ is closed in $\operatorname{Spec} R$ or that $\{Q \in \operatorname{Max\,Spec} R : R_Q$ is weakly F-regular$\}$ is closed in $\operatorname{Max\,Spec} R$, and let J be the radical ideal defining this set. Then J is multihomogeneous.*

(d) *If R is $\mathbb{N}$-graded and R_0 is a field, then R contains a homogeneous completely stable weak test element. If R is reduced, it contains a homogeneous completely stable test element.*

(e) *If R is a multigraded domain which has either a q'-weak test element or a completely stable q'-weak test element then there is such an element that is also multihomogeneous. In particular, if R has either a test element or a completely stable test element then it also has one that is multihomogeneous.*

Proof. The results in (a), (b), and (c) are immediate in case R contains an infinite field: the module or ideal in question is stable under the maps θ_α for all choices of α. In case R only contains a finite field K, consider a given element w in the module or ideal in question. After tensoring with a sufficiently large (automatically separable) finite field extension L of K we shall be able to choose sufficiently many distinct α to prove that all the graded pieces of w are in the corresponding ideal or module after tensoring with L, and it then follows that they were originally in it, since, by part a°) of Theorem 7.29 of [HH9], this ideal or module is simply obtained by applying $L\otimes_K$.

(e) is now immediate: if there is a nonzero element of a given sort, each of its nonzero graded pieces will serve. Because the ring is a domain, there is no issue about whether these pieces are in R°. The argument for (d) is similar. First note that an algebra of finite type over K always has a completely stable q'-weak test element, with $q' = 1$ if the ring is reduced. In the $\mathbb{N}$-graded case with $R_0 = K$, we simply note that if the set of forms in a homogeneous ideal I (which may be $\tau_{q'}(R)$ or $\hat{\tau}_{q'}(R)$) is covered by the minimal primes of R (which are homogeneous) then I is contained in one of them, by (4.1)(b). □

(4.3) Example. Let $R = K[X, Y]/(XY) = K[x, y]$, $\mathbb{N}^2$-graded so that the degree of $x^s y^t$ is (s, t). Let $z = x + y$. Then R_z is regular, so that

$z^t = x^t + y^t$ is a test element for sufficiently large t. However, the only multihomogeneous elements have the form x^r, y^s and, of these, only 1 is in R°. Thus, there is no multihomogeneous test element. There is also a problem if we $\mathbb{N}$-grade R so that elements of $K[x]$ have degree 0 and $\deg y = 1$. While the test ideal must be primary to (x, y) and so have the form (x^r, y^s), it does not contain any homogeneous element of R°. Here the problem is that $R_0 = K[x]$ is not a field.

Let R be a Noetherian $\mathbb{N}$-graded ring with $R_0 = K$, a field, and let $m = \bigoplus_{i>0} R_i$, the unique homogeneous maximal ideal of R. We conjecture that the following four conditions are equivalent:

(1) R is weakly F-regular.

(2) Every homogeneous ideal of R is tightly closed.

(3) Every graded submodule of a finitely generated graded R-module is tightly closed.

(4) R_m is weakly F-regular.

However, we cannot prove that they are all equivalent. We know that the last three are equivalent, but the first might be stronger. One problem is that we do not know that the weakly F-regular locus (even in the maximal spectrum) is open. When it is, all four conditions are equivalent, and this happens in the Gorenstein case. A second problem is that we do not know that tightly closed ideals remain tightly closed upon localization, even if the ideal is homogeneous and the multiplicative system consists of forms. (An even weaker version of localization would suffice: see Theorem 4.6 below.) Before proving what we do know to be true, we want to show that the weakly F-regular Gorenstein locus (which is the same as the F-regular Gorenstein locus) is open for a large class of rings. The proof depends on a thorough familiarity with the results of §§6 and 7 of [HH9], as well as knowledge of the notion of strong F-regularity introduced in [HH3] and also studied in §5 of [HH9].

(4.4) Theorem. *Let R be a local ring such that $R \to \widehat{R}$ has geometrically regular fibers (e.g., an excellent local ring). Let S be essentially of finite type over R. Then $\{P \in \operatorname{Spec} S : S_P$ is weakly F-regular and Gorenstein$\}$ is open in $\operatorname{Spec} S$.*

We first note:

(4.5) Lemma. *If $S \to T$ is a faithfully flat smooth map of Noetherian rings of characteristic p such that T is locally excellent and the weakly F-regular Gorenstein locus is open in $\operatorname{Spec} T$, then the weakly F-regular Gorenstein locus is open in $\operatorname{Spec} S$.*

Proof. In this argument we abbreviate "weakly F-regular Gorenstein" to WFRG. Let $Q_1, \cdots, Q_h$ be the minimal elements of the non-WFRG locus in $\operatorname{Spec} T$, so that $\bigcup_i V(Q_i)$ is the set of all primes Q in T such that T_Q is not WFRG. Let P_i be the contraction of Q_i to S. It will suffice to show that $\bigcup_i V(P_i)$ is the set of all primes P in S such that S_P is not WFRG. Note that when Q lies over P then $S_P \to T_Q$ is local, faithfully flat, and smooth. Since the closed fiber is Gorenstein, T_Q is Gorenstein iff S_P is. Since the map is faithfully flat, S_P is weakly F-regular if T_Q is, and since it is smooth as well, and T_Q is excellent, by Theorem 7.24 of [HH9] if S_P is weakly F-regular then T_Q is weakly F-regular. It follows that S_P fails to be WFRG iff T_Q fails to be WFRG. Thus, each of the primes P_i has the property of being in the complement of the WFRG locus for S. Let P be any prime in this complement. Since $S \to T$ is faithfully flat, we can choose $Q \in \operatorname{Spec} T$ lying over P. Then Q is in the complement of the WFRG locus in $\operatorname{Spec} T$, and so Q contains one of the Q_i. It follows that P contains the corresponding P_i. □

Proof of Theorem 4.4. Since $S \to \widehat{R} \otimes_R S$ is faithfully flat and smooth and since the second ring is essentially of finite type over $\widehat{R}$ and, consequently, excellent, it suffices, by (4.5), to prove the result for $\widehat{R}$. Henceforth, we assume that R is complete. Let P be a prime of S such that S_P is F-regular Gorenstein. Let K be a coefficient field for R, let Λ be a p-base for K, and for cofinite subsets $\Gamma \subseteq \Lambda$ let R^Γ be defined as in (6.11) of [HH9] with $S^\Gamma = R^\Gamma \otimes_R S$. By Lemma (6.13) of [HH9] we can choose Γ so that $P' = PS^\Gamma$ is prime. It follows that $S^\Gamma_{P'}$ is weakly F-regular (and Gorenstein, since the fibers of $R \to R^\Gamma$ are Gorenstein, by (6.6c) of [HH9]). Thus, it will suffice to prove that we can choose a Zariski neighborhood of P' consisting of primes P'' such that $S^\Gamma_{P''}$ is weakly F-regular and Gorenstein: the contractions of the P'' will share this property, since the map from the localization of S at one of these contractions to $S^\Gamma_{P''}$ is faithfully flat, and this will give the required neighborhood in S. Now R is module-finite over a complete regular ring A, and S^Γ is essentially of finite type over R^Γ and, hence, over A^Γ. It follows that the Gorenstein locus in S^Γ is open. Since S^Γ is module-finite over $(S^\Gamma)^p$, we know that the strongly F-regular locus in S^Γ is open (cf. Theorem (5.9b) of [HH9] and Theorem (3.3) of [HH3]). Thus, the strongly F-regular Gorenstein locus in S^Γ is Zariski open. However, this coincides with the weakly F-regular Gorenstein locus (cf. Theorem (5.5f) of [HH9] and Proposition (3.1) of [HH3]). □

(4.6) Theorem. *Let R be a Noetherian $\mathbb{N}$-graded ring with $R_0 = K$, a field of characteristic p. Consider the following conditions:*

(1) *R is weakly F-regular.*

(2) *Every graded submodule of a finitely generated graded module is tightly closed.*

(3) *Every homogeneous ideal of R is tightly closed.*

(4) *Every m-primary homogeneous ideal of R is tightly closed.*

(5) *R is Cohen-Macaulay, normal, and every irreducible m-primary homogeneous ideal is tightly closed.*

(6) *R_m is weakly F-regular.*

Then conditions (2)–(6) *are equivalent, and condition* (1) *implies them. Moreover, the equivalent conditions* (2)–(6) *imply that R is a normal domain.*

If R is Gorenstein, then conditions (1)–(6) *are equivalent.*

If for every homogeneous prime ideal P with $\dim R/P = 1$ *and for every homogeneous P-primary ideal Q, $S^{-1}Q$ is tightly closed in $S^{-1}R$, where S is the multiplicative system of all forms not in P, then* (1)–(6) *are also equivalent.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) is clear (that (4) $\Rightarrow$ (3) follows from the fact that every proper homogeneous ideal is an intersection of homogeneous ideals primary to the maximal ideal).

We next want to show that the equivalent conditions (3) and (4) imply that R is normal. The fact that (0) is tightly closed in R implies that R is reduced. Let W denote the multiplicative system consisting of all homogeneous elements of R that are not in any minimal prime of R. Let J denote the radical ideal defining the singular locus in R. Then J is homogeneous, and is not contained in any minimal prime of R. It follows that the forms in J cannot all be contained in the union of the minimal primes of R, by (3.1)(b). Thus, there is a form F in R° such that R_F is regular, and it follows that the normalization of R is a graded subring of R_F. Thus, if R is not normal there is a homogeneous element G/F^t of the normalization, where G is a form of R. But then the fact that G/F^t is integral over R implies that $G \in (F^tR)^* = F^tR$, and the fact $G \in F^tR$ implies that $G/F^t \in R$. This shows that R is normal. Since R is graded with $R_0 = K$, this in turn implies that R is a domain.

(4) implies that the ideal generated by a homogeneous system of parameters is tightly closed, and this remains true in R_m by Proposition 8.9 of [HH4]. Thus, by Theorem (4.3) of [HH9], R_m will be Cohen-Macaulay (since it is a domain, it is equidimensional). But since R_m is Cohen-Macaulay, R is Cohen-Macaulay (cf. [MRo] or [HoRa]). We therefore have that (4) $\Rightarrow$ (5).

Since the fact that R_m is weakly F-regular implies that R_m and, hence, R are Cohen-Macaulay, and since conditions (5) and (6) both imply that R_m is normal, to show that (5) and (6) are equivalent it will suffice to show that when R is Cohen-Macaulay there are m-primary irreducible homogeneous ideals cofinal with the powers of m: an m-primary ideal I is tightly closed in R iff IR_m is tightly closed in R_m by Proposition (8.9) of [HH4], and R_m is weakly F-regular iff there is a sequence $\{J_t\}_t$ of tightly closed mR_m-primary irreducible ideals cofinal with the powers of mR_m, by Proposition (8.8) of [HH4]. (There exist sequences of mR_m-primary irreducible ideals cofinal with the powers of mR_m because R_m is normal: this follows because normal local rings are approximately Gorenstein in the terminology of [Ho4]. See also the discussion in (8.6) of [HH4].)

Let $x_1, \cdots, x_d$ be a homogeneous system of parameters for R, and let ω be a graded canonical module for R. Let $u \in \omega$ be a homogeneous element representing a socle generator in $\omega/(x_1, \cdots, x_d)\omega$. Then, for every t, $(x_1 \cdots x_d)^{t-1}u$ represents a socle generator in $\omega_t = \omega/(x_1^t, \cdots, x_d^t)\omega$, and the image of u in ω_t spans a cyclic submodule of the form R/I_t, where $I_t = (x_1^t, \cdots, x_d^t)\omega :_R u$. Then I_t is an irreducible homogeneous m-primary ideal such that $(x_1 \cdots x_d)^{t-1}$ represents a socle generator in R/I_t. Moreover, $I_t u = (x_1^t, \cdots, x_d^t)\omega \cap Ru$, and it follows from the Artin-Rees lemma applied to $Ru \subseteq \omega$ that the ideals I_t are cofinal with the powers of $(x_1, \cdots, x_d)$ and, hence, with the powers of m. This completes the proof that (5) and (6) are equivalent.

We next want to show that (5) (or (6)) $\Rightarrow$ (2). First note that R_m is weakly F-regular and, hence, is a domain. Let N be a graded submodule of a finitely generated graded module M. As usual we may assume that $N = 0$, replacing M by M/N. If 0 is not tightly closed, we can choose a homogeneous element $x \in M$ in 0^*. Then $x \notin m^t M$ for $t \gg 0$, and so there will be a graded module killed by m^t in which 0 is not tightly closed. Let I be an m-primary homogeneous ideal contained in m^t (we know these exist when (5) holds). Then M can be embedded in a finite direct sum of copies of $(R/I)(s)$, where (s) indicates a grading shift. (Map a finite direct sum of copies of $(R/I)(s)$ onto the graded (R/I)-module $\mathrm{Hom}_R(M, R/I)$ and then apply $\mathrm{Hom}_R(\_, R/I)$ again.) This completes the proof that $(5) \Rightarrow (2)$.

We have shown that (2)–(6) are equivalent, and that (1) implies them. We have also shown that when the equivalent conditions (2)–(6) hold then R is a normal domain.

Now assume that the equivalent conditions (2)–(6) hold. We want to prove that R is weakly F-regular under a supplementary hypothesis.

First, suppose that R is Gorenstein. By Theorem 4.4, we know that the weakly F-regular Gorenstein locus is open. By Theorem 4.2, we know that the radical ideal J defining its complement (either in $\operatorname{Spec} R$ or in $\operatorname{Max}\operatorname{Spec} R$) is homogeneous. Thus, if R is not weakly F-regular, J is not the unit ideal, and we must have $J \subseteq m$. But this contradicts the hypothesis (6) that R_m is weakly F-regular.

Now suppose instead that, in addition to the equivalent conditions (2)–(6), we also assume that for every homogeneous prime P such that $\dim R/P = 1$ and for every homogeneous P-primary ideal Q, $S^{-1}Q$ is tightly closed in $S^{-1}R$, where S is the multiplicative system consisting of all forms in the complement of P. Again, we want to show that R is weakly F-regular. Notice that since R_m is weakly F-regular we know that R is a Cohen-Macaulay domain. (We also know that R is normal.)

Let n be any maximal ideal of R different from m. We shall prove that R_n is weakly F-regular. Let P be the prime ideal of R generated by all the homogeneous elements of n. Then $\dim R/P = 1$. Let S be the multiplicative system consisting of all forms in R not in P. No element of S is in n. Thus, n corresponds to a maximal ideal $N = nT$ of the $\mathbb{Z}$-graded ring $T = S^{-1}R$. Then $R_n \cong T_N$, and it will suffice to show that T_N is weakly F-regular.

Now, $T/PT \cong S^{-1}(R/P)$. Let h be the least positive integer in the group generated by the degrees of the nonzero forms in R/P. Let z be an element of $S^{-1}(R/P)$ of degree h. It is easy to check that the $\mathbb{Z}$-graded ring $S^{-1}(R/P)$ is $L[z, z^{-1}]$, where L is the degree 0 piece and is a field. It will suffice to show by Proposition (8.8) of [HH4] that there is a tightly closed irreducible N-primary ideal I_s in N^s for every fixed positive integer s.

We next observe that we can choose $Q \subseteq R$ homogeneous and P-primary such that QR_P is irreducible and contained in an arbitrarily high power of PR_P. First choose a system of parameters $x_1, \cdots, x_{d-1}$ for R_P consisting of forms in P. Let ω be a graded canonical module for R, so that ω_P is a canonical module for R_P. Let $(x^t) = (x_1^t, \cdots, x_{d-1}^t)$, and $y = \prod_{j=1}^{d-1} x_j$. Since $\operatorname{Hom}_R(R/P, \omega/(x)\omega)$ is graded, there will be a form u in $\omega/(x)\omega$ that represents the socle in $\omega_P/(x)\omega_P$. Now let Q_t be the expansion of $(x^t)\omega :_R y^{t-1}u$ with respect to the multiplicative system S. Since Q_t is homogeneous and contracted with respect to S, it is also contracted with respect to $R - P$ and, hence, P-primary: if there were a zero divisor on Q_t outside of P, there would have to be a homogeneous one, since the associated primes of Q_t are homogeneous. If we localize at P we have that $Q_tR_P = (x^t)\omega_P :_{R_P} y^{t-1}u$ and it follows that Q_tR_P is irreducible, PR_P-primary, and contained in arbitrarily high powers of PR_P as we let t become arbitrarily large.

Thus, given r, $Q_t \subseteq P^{(r)}$ for $t \gg 0$. Our next observation is that $P^{(r)}R_n \subseteq n^sR_n$ for given s and $r \gg 0$. (By Chevalley's theorem, it suffices to prove that $\bigcap_r P^{(r)}B = 0$, where $B = (R_n)\hat{}$. Choose a minimal prime $\mathfrak{P}$ of PB lying over PR_n. Then $P^{(r)}R_n = (PR_n)^{(r)} \subseteq \mathfrak{P}^{(r)}$. Now $\bigcap_r \mathfrak{P}^{(r)}B_{\mathfrak{P}} = 0$, and so $\bigcap_r \mathfrak{P}^{(r)}$ is killed by an element of $B - \mathfrak{P}$. Since R is normal (and excellent) B is a domain, and so $\bigcap_r \mathfrak{P}^{(r)} = (0)$. This shows that $\bigcap_r P^{(r)}B = (0)$.)

We have now shown that one can construct a homogeneous P-primary ideal $Q \subseteq n^s$ for a given s such that QR_P is irreducible. Now, nT/PT is a maximal ideal of $T/PT \cong L[z, z^{-1}]$. Choose $\beta \in T$ representing a generator of nT/PT. Note that the image of β in $L[z, z^{-1}]$ must be an inhomogeneous polynomial in z (since the nonzero homogeneous elements are all units). In fact, we can assume that each nonzero homogeneous piece of β is outside P and, hence, invertible in T.

Since QR_P is irreducible, $\mathrm{Hom}_T(T/PT, T/QT) \cong \mathrm{Ann}_{T/QT}P$ is a rank one (torsion-free, since Q is P-primary) module over $T/PT \cong L[z, z^{-1}]$. Thus, $\mathrm{Ann}_{T/QT}P$ is graded and free on one generator over $L[z, z^{-1}]$. Let $F \in T$ be a form that represents this generator.

We claim that $QT+\beta^rT$ is an irreducible N-primary ideal for all values of $r \geq 1$, and that the socle generator is represented by $\beta^{r-1}F$. (Once Q is chosen so that $QT \subseteq N^s$ we can take $r = s$ and get $QT + \beta^rT \subseteq N^s$.) In fact, since Q is P-primary, T_N/QT_N is a one-dimensional Cohen-Macaulay ring, and since $N = PT + \beta T$, β is a parameter. It will suffice to show that T_N/QT_N is Gorenstein and that the image of F generates the socle modulo β. For the first we note that T/QT is Gorenstein: if it were not, the non-Gorenstein locus would be defined by a proper homogeneous radical ideal. Since every form not in PT/QT is invertible, such an ideal must be contained in PT/QT, and it would follow that $(T/QT)_{PT} \cong (R/Q)_P$ is not Gorenstein, a contradiction (since this ring is zero-dimensional and Q_{RP} is irreducible in R_P). Thus, T/QT is Gorenstein. Since β is not a zerodivisor in T/QT, it is not a zerodivisor on $\mathrm{Ann}_{T/QT}P = F(T/QT) \cong T/PT$. Thus, we have a short exact sequence

$0 \to F(T/QT) \to T/QT \to T/(Q+FT) \to 0$. Since β is not a zerodivisor on the rightmost module, the sequence remains exact when we apply $\otimes_T T/\beta T$. We can then localize at N. The leftmost term is then isomorphic with $T_N/(PT_N+\beta T_N) \cong T_N/NT_N$ and injects into $T_N/(QT_N+\beta T_N)$. This shows that the image of F generates a copy of the residue field in $(T/Q)_N/\beta(T/Q)_N$, and this must be the socle.

It remains to prove that each of the ideals $QT+\beta^r T$ is tightly closed (localizing at the maximal ideal N will not affect the issue). Since T is a $\mathbb{Z}$-graded domain finitely generated over K, there is a homogeneous test element c, which we may assume is in R. Since $\beta^{r-1}F$ represents the socle, we need only show that it cannot be in the tight closure of $QT+\beta^r T$ in T.

Suppose that $c(\beta^{r-1}F)^q \in Q^{[q]}T+\beta^{qr}T$ for all q. This yields $c\beta^{(r-1)q}F^q = v+\beta^{qr}\theta$ where $v \in Q^{[q]}$ and $\theta \in T$. Then $\beta^{qr}\theta \in Q^{[q]}T+cF^qT$. Now, β cannot be a zerodivisor on any homogeneous ideal of T, since it would then be contained in an associated prime of it, which would be homogeneous, and the nonzero graded pieces of β are units. It follows that $\theta \in Q^{[q]}+cF^qT$, say $\theta = w+cF^q\theta'$ where $w \in Q^{[q]}T$ and $\theta' \in T$. But then we obtain that $\beta^{(r-1)q}cF^q = v'+\beta^{qr}cF^q\theta'$ where $v' = v+\beta^{qr}w \in Q^{[q]}T$. We then have that $\beta^{(r-1)q}(1-\beta^q\theta')(cF^q) = v' \in Q^{[q]}T$ and, again, since β cannot be a zerodivisor on the homogeneous ideal $Q^{[q]}T$, we have that $(1-\beta^q\theta')cF^q \in Q^{[q]}T$ or that $1-\beta^q\theta' \in Q^{[q]}T :_T cF^q$.

This implies that β is invertible modulo $Q^{[q]}T :_T cF^q$ for all q, and so $Q^{[q]} : cF^q$ cannot be contained in PT, since β is not invertible modulo PT. But the homogeneous ideal $Q^{[q]}T : cF^q$ cannot contain a form outside P unless it contains a unit, since all forms of T not in PT are invertible. Thus, $cF^q \in Q^{[q]}T$ for all q, and this means that F is in the tight closure of QT, contradicting the hypothesis. $\square$

We can prove a better result for F-rationality: recall that a Noetherian ring R of characteristic p is *F-rational* if every ideal generated by parameters is tightly closed. We refer the reader to §4 of [HH9] and to [FeW] for more information. If R is a homomorphic image of a Cohen-Macaulay ring then this condition implies that R is Cohen-Macaulay and normal. Since we shall only be discussing finitely generated algebras over a field here, all F-rational rings here will be Cohen-Macaulay and normal.

The condition of F-rationality is local on the maximal ideals of R. Moreover, if a local ring R is equidimensional, then it is F-rational if (and, of course, only if) the ideal generated by a single system of parameters is tightly closed. In the Gorenstein case, F-rationality, weak F-regularity and

F-regularity are equivalent. All of these results may be found in §4 of [HH9].

Remark. It has recently been shown [Ve1] that the F-rational locus is open in a ring of characteristic p finitely generated over an excellent ring, and that if $h: R \to S$ is a flat homomorphism with geometrically regular fibers, R, S are locally excellent, and R is F-rational of characteristic p, then S is F-rational. Using these facts it is possible to prove that the conditions (1)–(4) given in (4.7) below are equivalent without the hypotheses (a) and (b). The point is that, after enlarging the field if necessary, one sees as usual that the radical ideal defining the complement of the F-rational locus must be homogeneous, from which (4) $\Rightarrow$ (1) follows at once. However, we thought it worthwhile to record an argument that does not need the machinery introduced in [Ve1].

We want to thank the referee for pointing out an error in an earlier treatment of (4.7).

(4.7) Theorem. *Let R be a Noetherian $\mathbb{N}$-graded ring of characteristic p such that $R_0 = K$ is a field. Let $m = \bigoplus_{i>0} R_i$ be the unique homogeneous maximal ideal.*

Suppose also that

(a) *K is perfect and that*

(b) *R is generated over K by its forms of degree one.*

Then the following conditions are equivalent:

(1) *R is F-rational.*

(2) *Every ideal generated by part of a homogeneous system of parameters is tightly closed.*

(3) *R is equidimensional and the ideal generated by a single homogeneous system of parameters is tightly closed.*

(4) *R_m is F-rational.*

Moreover, the implications (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) *hold without the hypotheses* (a), (b).

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) follow the same lines as in the proof of Theorem 4.6 and are omitted. We next observe that (4) $\Rightarrow$ (2). Assume (4), and suppose that I is an ideal generated by part of a homogeneous system of parameters. Then I^*/I must be graded and, hence, supported at m. But for R-sequences tight closure commutes with localization, by Theorem 4.5 of [HH9], and so $(I^*/I)_m = (IR_m)^*/IR_m = 0$, for IR_m is tightly closed. Thus, $I = I^*$.

Now assume that the equivalent conditions (3) and (4) hold and also assume that K is perfect and that R is $K[R_1]$. We must prove (1). We want to replace R by $L \otimes_K R$, where L is an algebraic closure of R. The ideal generated by a homogeneous system of parameters in R is tightly closed, and this remains true in $L \otimes_K R$ by part (ii) of Theorem (7.29a) of [HH9]. If we can show that $L \otimes_K R$ is F-rational, then the same will follow for R. (For every maximal ideal n of R containing a system of parameters z_j for R_n, there is a maximal ideal $\mathscr{n}$ of S lying over n, and the z_j are also a system of parameters in $S_{\mathscr{n}}$. From the fact that the ideal (z_j) is tightly closed in $S_{\mathscr{n}}$ we can conclude that it is tightly closed in R_n, since the map $R_n \to S_{\mathscr{n}}$ is faithfully flat.) We henceforth assume that K is algebraically closed.

Now let n be any maximal ideal of R. As in the proof of (4.6) let P be the prime ideal generated by all the homogeneous elements of n, so that $\operatorname{ht} P = \dim R - 1 = d - 1$, let S be the multiplicative system of R generated by all forms not in P, and let T be the $\mathbb{Z}$-graded ring $S^{-1}R$. However, now we have that $L = K$, and R/n is also K. As in the proof of (4.6) we can choose $\beta \in T$ such that $N = nT$ is generated by β over PT. However, because K is algebraically closed we may now assume that β has the form $z - \lambda$, where $\lambda \in K$ and z is a form of $S^{-1}R$ that maps to a form of smallest positive degree, namely, to a form of degree one (because of assumption (b)) in $S^{-1}(R/P) \cong K[\zeta, \zeta^{-1}]$ (where we have also written ζ for the image of z modulo $PS^{-1}R$). Then $R_n \cong T_N$ and it will suffice to show that some system of parameters for NT_N generates a tightly closed ideal. Choose forms $x_1, \cdots, x_{d-1} \in P$ which give a system of parameters for R_P and generate an ideal of height $d-1$ in R. Let $(x) = (x_1, \cdots, x_{d-1})$. $(x)R$ may have some associated primes other than P, but they will contain forms not in P. Thus, $\operatorname{Rad}(x)T = PT$ and $(x)T$ is primary to PT. It follows that $N = \operatorname{Rad}((x)T + \beta T)$ and that $x_1, \cdots, x_{d-1}, \beta$ is a system of parameters for T_N. To complete the proof, it will suffice to show that $(x)T + \beta T$ is tightly closed in T.

Now $\operatorname{Hom}_T(T/PT, T/(x)T) \cong \operatorname{Ann}_{T/(x)T} P$ is a torsion-free graded module of rank, say, r, over $T/PT \cong K[\zeta, \zeta^{-1}]$, and so has a free basis over T/PT generated by elements represented by forms $F_1, \cdots, F_r$ of T. Since $\deg z = 1$, we can multiply each F_j by a suitable power of z and so assume that $\deg F_j = 0$ for all j. Here, r is the type of the zero-dimensional ring $T_{PT}/(x)T_{PT} \cong R_P/(x)R_P$. Since the locus where the type is strictly bigger than r is closed and defined by a homogeneous ideal of T, this locus is empty, for the ideal cannot be contained in P. It follows that the type of T_N is at most r. On the other hand, as in the proof of (4.6) we see that β is not a zerodivisor on $(T/(x)T)/\operatorname{Ann}_{T/(x)T} P$, and it follows that the (T/N)-vector space $(T/\beta T) \otimes_T \operatorname{Ann}_{T/(x)T} P$, which has dimension r, injects into $T/((x)T + \beta T)$. Thus, we may identify the socle in $T/((x)T + \beta T)$ with the r-dimensional (T/N)-vector space generated by the images of the F_i. Since $\beta = z - \lambda$, we see that no element of $\sum_j KF_j$ maps to 0 when we kill $\beta(\mathrm{Ann}_{T/(x)T} P)$. It follows that if there is an element in the tight closure of $x(T) + \beta T$ not in $(x)T + \beta T$, then there is such an element in the socle $(T/\beta T) \otimes_T \mathrm{Ann}_{T/(x)T} P$, and it is represented by a nonzero form $F \in \sum_j KF_j$. As before, there is a homogeneous test element c, and we must have $cF^q \in (x^q)T + \beta^q T$. As in the earlier argument, β is not a zerodivisor on $(cF^q) + (x^q)T$, and it follows that the coefficient of β^q is in $cF^q T + (x^q)T$. This yields $cF^q = v + \beta^q cF^q \theta$ where $v \in (x^q)T$ and $\theta \in T$. This yields $(1 - \beta^q \theta) \in (x^q)T :_T cF^q$ as in the earlier argument, and we conclude that the homogeneous ideal $(x^q)T :_T cF^q$ is not contained in P, as before. This shows that $cF^q \in (x^q)T$, and so $F \in ((x)T)^*$. But $(x)R$ is a tightly closed ideal generated by an R-sequence, and so remains tightly closed when we localize, by Theorem (4.5) of [HH4]. It follows that $F \in (x)T$, and this a contradiction. □

(4.8) Theorem. *Let K be a field, let $x_1, \cdots, x_n$ be indeterminates over K, and give $T = K[x_j, x_j^{-1}]$ a $\mathbb{Z}^n$-grading such that*

$$\deg \prod_j x_j^{a_j} = (a_1, \cdots, a_n).$$

Let R be a subring of T generated by finitely many monomials in the x's and let I be a multihomogeneous ideal of R, generated by monomials $\mu_1, \cdots, \mu_s$. Then I^* is the sum of the ideals $(\mu_j R)^*$, and $(\mu_j R)^*$ is identical with the integral closure of $\mu_j R$ in R.

Proof. We already know that there is a multihomogeneous test element $c \neq 0$ and that I^* is multihomogeneous, from (4.2e) and (4.2b), respectively. Let u be a form in I^*. Then for all q we have that $cu^q \in I^{[q]} = (\mu_1^q, \cdots, \mu_s^q)T$. But in T, a form is in an ideal generated by forms if and only if it is a multiple of one of the generators. (The key point is that each multigraded piece of T is a vector space over K of dimension at most one.) Since there are only finitely many μ_j it follows that there exist j and infinitely many values of q such that $cu^q \in \mu_j^q T$. But this implies that $u \in (\mu_j T)^*$ by Lemma (8.16) of [HH4]. Since T is a domain, $(\mu_j T)^*$ is the same as the integral closure of $\mu_j T$ by Corollary (5.8) of [HH4]. □

Note that §7 contains results giving criteria for F-rationality of graded rings that are useful for establishing F-rationality in specific examples.

5. Phantom extensions, big Cohen-Macaulay modules and splitting in module-finite extensions

In this section we introduce the notion of a *phantom extension* of modules. We use it both to give a recasting of the proof of the existence of big Cohen-Macaulay modules in characteristic p and as a starting point for an investigation of when the contraction of an expanded ideal is within the tight closure of the original ideal (in many cases, one has a result of this type even for the contraction of the tight closure of the expansion of the ideal). Results of this type lead to splitting theorems when the original ring is weakly F-regular under various hypotheses on the ring extension.

Phantom extensions are extensions that are "almost zero" in a tight closure sense. Their relevance to the existence of big Cohen-Macaulay modules is simply this: given a phantom extension of a local ring R (viewing R as an R-module), say $R \to M$, one can show in many cases that the image of the generator of R is a basic element in M (see Proposition (5.14)). But one can also show that when one makes a "modification" of M of the type needed in the proof of the existence of big Cohen-Macaulay modules, say $M \to M'$, the composite map $R \to M \to M'$ is still a phantom extension. This enables one to give a complete proof of the existence of big Cohen-Macaulay modules easily. The details are given in (5.15) and (5.16).

(5.1) Notation and preliminaries. Throughout this section (excepting (5.5) and (5.6)) let R be a Noetherian ring of characteristic p and let M, N, Q denote finitely generated R-modules. Let $\alpha\colon N \to M$ be a map with cokernel Q, and let $\beta\colon M \to Q$ be the induced surjection. Eventually we shall specialize our hypotheses to the case where α is injective, $N = R$, and R is reduced, but for the present we retain greater generality. In this section we shall study a weakening, given in Definition 5.2 below, of the condition that the map α split (i.e. that there exist $\alpha'\colon M \to N$ such that $\alpha' \circ \alpha = \mathrm{id}_N$): when this condition holds, we shall say that the sequence $N \to M \to Q \to 0$ is a *phantom extension* of Q by N.

By abuse of terminology we shall also say that α is a *phantom extension* or that M is a *phantom extension* of N: the context should make clear what is meant. We shall use the theory of phantom extensions to prove that if R is weakly F-regular, then R is a direct summand of its module-finite extensions, as well as related generalizations. We shall also use this notion to sketch a proof of the existence of big Cohen-Macaulay modules in characteristic p. This argument is, in essence, a recasting of the original argument of [Ho3] in the framework of tight closure. In fact, one might argue that the theory of tight closure really began with that particular argument. However, we want to make the connection explicit, because the new version of the proof shows that a sufficiently good theory of tight closure in mixed characteristic would yield the existence of big Cohen-Macaulay modules: this is a long-standing open question. We note that the direct treatment of each of the local homological conjectures given in §10 of [HH4] and §§4 and 6 of [HH8] (e.g., the phantom intersection theorem) via tight closure techniques turned out to be eminently worthwhile.

(5.2) Definition. With notation as in (5.1) above, we shall say that $N \xrightarrow{\alpha} M \xrightarrow{\beta} Q \to 0$ is a *phantom extension* if there exists an element $c \in R^{\circ}$ such that for all $e \gg 0$, there is a map $\gamma_e: \mathbf{F}^e(M) \to \mathbf{F}^e(N)$ such that $\gamma_e \circ \mathbf{F}^e(\alpha) = c(\mathrm{id}_{\mathbf{F}^e(N)})$.

(5.3) Remark. When α is injective, the sequence

$$0 \longrightarrow N \xrightarrow{\alpha} M \xrightarrow{\beta} Q \longrightarrow 0$$

represents an element ε of $\mathrm{Ext}^1_R(Q, N)$. This Ext may be viewed as part of the cohomology of the complex $\mathrm{Hom}_R(P_., N)$, where $P_.$ is a projective resolution of Q by finitely generated projectives (with the augmentation Q dropped). We shall show that under certain conditions (see (5.8) and (5.13) below) the extension is phantom in the sense of Definition 5.2 if and only if ε, viewed as an element in the homology of $\mathrm{Hom}_R(P_., N)$, is a phantom element.

(5.4) Remark. Suppose that R is reduced. Then the algebra map $R \xrightarrow{\mathbf{F}^e} R$ may be identified with the inclusion map $R \to R^{1/q}$, where $q = p^e$, as usual. The condition that $N \xrightarrow{\alpha} M \xrightarrow{\beta} Q \to 0$ be phantom may then be rephrased as follows: there is an element $c \in R^{\circ}$ such that, for all $e \gg 0$, there is an $R^{1/q}$-linear map $\gamma_e: R^{1/q} \otimes_R M \to R^{1/q} \otimes_R N$ such that $\gamma_e \circ (R^{1/q} \otimes_R \alpha) = c^{1/q}(\mathrm{id}_{R^{1/q} \otimes N})$.

(5.5) Discussion. For the moment (in (5.5) and (5.6)) we suspend all finiteness conditions and characteristic conditions on rings and modules. First note that if $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} Q \to 0$ is an exact sequence and

$$\cdots \to P_2 \to P_1 \xrightarrow{d} P_0 \to Q \to 0$$

is the beginning of a projective resolution of Q, then we may fill in vertical arrows to obtain:

$$(\#)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & Q & \longrightarrow & 0 \\ \uparrow & & \uparrow{\scriptstyle\phi} & & \uparrow & & \uparrow{\scriptstyle\mathrm{id}} & & \\ \longrightarrow P_2 & \longrightarrow & P_1 & \xrightarrow{d} & P_0 & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

In the Yoneda correspondence, the exact sequence $(*)$ $0 \to N \to M \to Q \to 0$ corresponds to the element of $H^1(\mathrm{Hom}_R(P_.\,, N))$ represented by ϕ. On the other hand, given a cocycle ϕ (the cocycle condition simply means that ϕ vanishes on the image of P_2), an exact sequence can be constructed by letting M be the pushout $N\amalg_{P_1} P_0 \cong (N\oplus P_0)/\{\phi(u)-d(u) : u \in P_1\}$, and using the canonical injection $N \to M$ and the fact that $M/\operatorname{Im} N \cong P_0/\phi(P_1) \cong Q$.

Note that if we tensor with an R-algebra S, we still have a projective resolution $G_.$ for $S \otimes_R Q$ that begins

$$\cdots \longrightarrow G_2 \longrightarrow S \otimes_R P_1 \xrightarrow{S\otimes d} S \otimes_R P_0 \longrightarrow S \otimes Q \longrightarrow 0,$$

although we will not have, in general, that $S \otimes_R P_2$ maps onto the kernel of $S \otimes d$; i.e. we cannot necessarily take $G_2 = S \otimes_R P_2$. Note that an element in $\mathrm{Hom}_S(S \otimes_R P_1, S \otimes_R N)$ is a cocycle iff it vanishes on $\mathrm{Im}(G_2 \to S \otimes_R P_1) = \operatorname{Ker} S \otimes d$, while an element is a coboundary iff it is in the image of the map $\mathrm{Hom}_S(S \otimes_R P_0, S \otimes_R N) \to \mathrm{Hom}_S(S \otimes_R P_1, S \otimes_R N)$ induced by $S \otimes d$.

Note also that if $\phi \in \mathrm{Hom}_R(P_1, N)$ is the map obtained from the sequence $(*)$, $S \otimes_R \phi$ will not necessarily be a cocycle in $\mathrm{Hom}_S(S \otimes_R P_1, S \otimes_R N)$. It turns out to be a cocycle iff tensoring $(*)$ with S produces an exact sequence, in which case it represents the extension given by the new sequence.

(5.6) Lemma. *Let notation be as in* (5.5).

(a) *For each* $c \in R$, $c\phi$ *is a coboundary in* $H^1(\mathrm{Hom}_R(P_.\,, N))$ *if and only if there is a map* $\gamma\colon M \to N$ *such that* $\gamma\alpha = c(\mathrm{id}_N)$.

(b) *An arbitrary R-linear map* $\psi\colon P_1 \to N$ *yields a sequence*

$$0 \to N \to N\amalg_{P_1} P_0 \to Q \to 0$$

which is exact except possibly at N, *and is exact at* N *if and only if* ψ *is a cocycle.*

(c) *Suppose that* $(*)$ $0 \to N \to M \to Q \to 0$ *is exact and corresponds to the element of* $H^1(\mathrm{Hom}_R(P_.\,, N)) \cong \mathrm{Ext}^1_R(Q, N)$ *represented by* $\phi\colon P_1 \to N$. *The sequence remains exact upon tensoring with the R-algebra S if and only if* $S \otimes_R \phi \in \mathrm{Hom}_S(S \otimes_R P_1, S \otimes_R N)$ *is a* 1*-cocycle in*

$\mathrm{Hom}_S(G_\cdot, S\otimes_R N)$, *in which case* $S\otimes_R \phi$ *represents the extension over* S *given by the sequence* $0\to S\otimes_R N\to S\otimes_R M\to S\otimes_R Q\to 0$.

Proof. (a) To give a map $\gamma: M=(N\oplus P_0)/\{c\phi(u)-d(u): u\in P_1\}\to N$ such that $\gamma\circ\alpha=c(\mathrm{id}_N)$ is equivalent to giving a map $\lambda: P_0\to N$ such that the map $(c(\mathrm{id}_N)\oplus\lambda): (N\oplus P_0)\to N$ kills $\{\phi(u)-d(u): u\in P_1\}$, i.e. such that $c\phi(u)=(\lambda\circ d)(u)$ for all $u\in P_1$. But λ satisfies this condition iff $c\phi$ is its coboundary.

(b) It is straightforward to verify that there is always an exact sequence $N\to N\amalg_{P_1}P_0\to Q\to 0$, where $N\amalg_{P_1}P_0\cong(N\oplus P_0)/\{\phi(u)-d(u): u\in P_1\}$. The injectivity is equivalent to the assertion that $\{\phi(u)-d(u): u\in P_1\}$ meets $N\oplus 0$ in $0\oplus 0$, i.e. that $d(u)=0\Rightarrow\phi(u)=0$. Since $\mathrm{Ker}\, d=\mathrm{Im}\, P_2$, this simply says that ϕ is a cocycle.

(c) Upon tensoring the diagram (#) above (used to construct a representative ϕ for the element of Ext corresponding to the sequence $0\to N\xrightarrow{\alpha}M\xrightarrow{\beta}Q\to 0$) with S we get the diagram:

$$(\#\#)\qquad \begin{array}{ccccccccc} \longrightarrow & S\otimes N & \xrightarrow{S\otimes\alpha} & S\otimes M & \xrightarrow{S\otimes\beta} & S\otimes Q & \longrightarrow & 0 \\ & \uparrow{\scriptstyle S\otimes\phi} & & \uparrow & & \uparrow{\scriptstyle \mathrm{id}} & & \\ \longrightarrow & S\otimes P_1 & \xrightarrow{S\otimes d} & S\otimes P_0 & \longrightarrow & S\otimes Q & \longrightarrow & 0 \end{array}$$

which shows that if $S\otimes\alpha$ is injective, then the element of Ext corresponding to the first row is represented by $S\otimes\phi$, and then $S\otimes\phi$ must be a cocycle by part (b) just above (applied over S). On the other hand, if $S\otimes\phi$ is a cocycle it will represent the element of $\mathrm{Ext}^1_S(S\otimes Q, S\otimes N)$ determined by the canonical injection of $S\otimes N$ into the pushout $S\otimes N\amalg_{S\otimes P_1}S\otimes P_0$, and the right exactness of tensor (apply $S\otimes$ to $P_1\xrightarrow{\eta}N\times P_0\to N\amalg_{P_1}P_0\to 0$, where $\eta=(\phi, -d)$) implies that this extension is isomorphic with the result of applying $S\otimes$ to the extension $0\to N\xrightarrow{\alpha}M\xrightarrow{\beta}Q\to 0$ with which we started. □

We now restore our characteristic p and finiteness assumptions. We next observe:

(5.7) Proposition. *Let* $N\xrightarrow{\alpha}M\xrightarrow{\beta}Q\to 0$ *be a phantom extension.*

(a) *The result of tensoring with any Noetherian* R*-algebra* S *such that* R° *maps into* S° *is a phantom extension. In particular, we may choose* S *to be any localization of* R *or, more generally, any flat* R*-algebra.*

(b) $\mathbf{F}^e(N)\to\mathbf{F}^e(M)\to\mathbf{F}^e(Q)\to 0$ *is a phantom extension for all* e.

(c) *If* I *is an ideal of* R, *then* $\alpha^{-1}(IM)\subseteq(IN)^*_N$. *In particular, if* $N=R$, $\alpha^{-1}(IM)\subseteq I^*$.

(d) *The kernel of the map* $\mathbf{F}^e(N) \to \mathbf{F}^e(M)$ *is in the tight closure of* 0 *in* $\mathbf{F}^e(N)$ *for all* e.

(e) *If* $\alpha(N) \subseteq M_0 \subseteq M$, *then* $N \to M_0 \to M_0/\alpha(N) \to 0$ *is a phantom extension.*

(f) *If* $M \xrightarrow{\alpha'} L \to T \to 0$ *is also a phantom extension, then* $N \xrightarrow{\alpha' \circ \alpha} L \to \operatorname{Coker}(\alpha' \circ \alpha) \to 0$ *is a phantom extension.*

Proof. (a) We use a subscript on the Frobenius functor $\mathbf{F}$ to indicate when we are applying it over S; i.e. we write $\mathbf{F}_S$. Suppose we have a map $\beta_e : \mathbf{F}^e(M) \to \mathbf{F}^e(N)$ such that $\beta_e \circ \mathbf{F}^e(\alpha) = c(\mathrm{id}_{\mathbf{F}^e(N)})$. By the associativity of tensor, the functors sending the R-module W to $\mathbf{F}_S^e(S \otimes_R W)$ and to $S \otimes_R \mathbf{F}^e(W)$ are canonically isomorphic. Let c' be the image of c in S. With this identification, we see that $(S \otimes_R \beta_e) \circ (\mathbf{F}_S^e(S \otimes_R \alpha)) = c'(\mathrm{id}_{\mathbf{F}_S^e(S \otimes N)})$.

(b) This now follows from (a) with $S = R$ and the structural homomorphism $\mathbf{F}^e : R \to R$.

(c) Suppose that $u \in N$ and $\alpha(u) \in IM$. Then $\mathbf{F}^e(\alpha)(u^q) \in I^{[q]}\mathbf{F}^e(M)$ for all e, and for large e we may apply γ_e as in the definition of phantom extension to both sides to obtain $cu^q \in I^{[q]}\mathbf{F}^e(N)$ for all $e \gg 0$ with $c \in R^\circ$. Since $I^{[q]}\mathbf{F}^e(N)$ is the image of $\mathbf{F}^e(IN)$ in $\mathbf{F}^e(N)$, the result follows.

(d) By part (b), $\mathbf{F}^e(N) \to \mathbf{F}^e(M) \to \mathbf{F}^e(Q) \to 0$ is a phantom extension: now apply part (c) with $I = 0$.

(e) For each $e \gg 0$, replace γ_e by its composition with $\mathbf{F}^e(M_0) \to \mathbf{F}^e(M)$.

(f) That α (respectively, α') is phantom yields the existence of a fixed $c \in R^\circ$ (respectively, $c' \in R^\circ$) and, for all $e \gg 0$, a map $\gamma_e : \mathbf{F}^e(M) \to \mathbf{F}^e(N)$ (respectively, $\delta_e : \mathbf{F}^e(L) \to \mathbf{F}^e(M)$) such that $\gamma_e \circ \mathbf{F}^e(\alpha) = c(\mathrm{id}_{\mathbf{F}^e(N)})$ (respectively, $\delta_e \circ \mathbf{F}^e(\alpha') = c'(\mathrm{id}_{\mathbf{F}^e(M)})$). But then $(\gamma_e \circ \delta_e) \circ (\mathbf{F}^e(\alpha' \circ \alpha)) = \gamma_e \circ (\delta_e \circ \mathbf{F}^e(\alpha')) \circ \mathbf{F}^e(\alpha) = \gamma_e \circ (c' \,\mathrm{id}_{\mathbf{F}^e(M)}) \circ \mathbf{F}^e(\alpha) = c'(\gamma_e \circ \mathbf{F}^e(\alpha)) = c'c(\mathrm{id}_{\mathbf{F}^e(N)})$ for all $e \gg 0$. $\square$

Parts (e) and (f) are both analogues of facts that are obvious for direct summands: e.g., (f) is an analogue of the fact that a direct summand of a direct summand is a direct summand. One must be quite careful however: there are obvious properties of split maps that do not carry over to the phantom case. We note particularly that if $N \xrightarrow{\alpha} M \to Q \to 0$ is a phantom extension over R and one restricts scalars to A, where $A \to R$ is a module-finite extension, the same sequence need not be a phantom extension when considered over A! See Example 5.37, where $N = R$, A is regular, and the map $\mathbf{F}^e(\alpha)$ is injective for all e.

(5.8) Proposition. *Let* $(*)$ $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} Q \to 0$ *be an exact sequence such that* $\mathbf{F}^e(\alpha)$ *is injective for all* e. *Then the following two statements are equivalent:*

(a) $(*)$ *is a phantom extension.*

(b) *If* $P_.$ *is a resolution of* Q *by finitely generated projective modules, then the element* ε *corresponding to the sequence* $(*)$ *in* $\mathrm{Ext}^1_R(Q, N)$ *corresponds to a phantom element in* $H^1(\mathrm{Hom}_R(P_. , N))$ *(i.e. a cocycle that is in the tight closure of the coboundaries in* $\mathrm{Hom}_R(P_1, N))$.

Proof. Let $\phi \in \mathrm{Hom}_R(P_1, N)$ be constructed as in the diagram labeled (#) in (5.5), so that it represents the element of $\mathrm{Ext}^1_R(Q, N)$ corresponding to the given sequence. Fix e and let S denote R viewed as an R-algebra via the structural homomorphism $\mathbf{F}^e$. Since P_1 is a finitely generated projective module, there is a canonical identification of $S \otimes_R \mathrm{Hom}_R(P_1, N)$ with $\mathrm{Hom}_S(S \otimes_R P_1, S \otimes_R N)$. When we think of the Hom as an S-module and remember that $S = R$, we obtain $\mathbf{F}^e(\mathrm{Hom}_R(P_1, N))$. But then $\mathbf{F}^e(\phi) = \mathrm{id}_S \otimes_R \phi$ may be thought of as ϕ^q, where $q = p^e$, and so by part (c) of (5.6), ϕ^q corresponds to the sequence obtained by applying $S \otimes_R$ to $(*)$. By part (a), there exists a map γ_e such that $\gamma_e \circ \mathbf{F}^e(\alpha) = c(\mathrm{id}_{\mathbf{F}^e(N)})$ iff $c\phi^q$ is a coboundary, i.e. is in the image of the map $S \otimes_R \mathrm{Hom}_R(P_0, N) \to S \otimes_R \mathrm{Hom}_R(P_1, N)$. Let B be the image of $\mathrm{Hom}_R(P_0, N) \to \mathrm{Hom}_R(P_1, N)$. By the right exactness of tensor product, the module of coboundaries is the same as the image of $S \otimes_R B$ in $S \otimes_R \mathrm{Hom}_R(P_1, N)$. Thus, the defining condition for $(*)$ to be a phantom extension holds for fixed $c \in R^\circ$ and all $e \gg 0$ iff $c\phi^q \in \mathrm{Im}(\mathbf{F}^e(B) \to \mathbf{F}^e(\mathrm{Hom}_R(P_1, N)))$ for all $e \gg 0$, which is precisely the condition for ϕ to be in the tight closure of B in $\mathrm{Hom}_R(P_1, N)$, i.e. to be a phantom element in $H^1(\mathrm{Hom}_R(P_\bullet, N))$. □

(5.9) Corollary. *Let* $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} Q \to 0$ *be an exact sequence such that* $\mathbf{F}^e(\alpha)$ *is injective for all* e. *Suppose also that the extension is phantom. If* R *is weakly F-regular and, in particular, if* R *is regular, then the sequence splits.*

Proof. We may apply (5.8). Since the element in $\mathrm{Ext}^1_R(Q, N)$ is phantom and R is weakly F-regular, the element is 0, and the extension is therefore trivial. □

(5.10) Remark. Of course, any short exact sequence that is split, i.e., that represents a trivial extension, is a phantom extension.

(5.11) Remark. For each of the variant notions of tight closure discussed in §10 of [HH4], there is a corresponding variant notion of phantom extension. One simply varies the quantification in Definition (5.2) (concerning which c and e must work in the tests) in the obvious way. One obtains an analogue of (5.8) for each variant notion. We have decided to avoid technicalities and treat only the main case here. However, there is definite motivation for considering the variant notions. For example, suppose we consider the variant notion for Cohen-Macaulay tight closure. Then every element c such that R_c is C-M will have a power that must work in phantom extension tests. If one tensors with a weakly F-regular R-algebra S which *merely satisfies the condition that the image of* c *is not zero*, then one can conclude that the sequence obtained after tensoring still gives a phantom extension. One may then be able to bring Corollary (5.9) to bear and conclude that the sequence splits after tensoring.

(5.12) Proposition. *Let* R *be reduced and* $\alpha\colon R \to M$ *injective. Then* $\mathbf{F}^e(\alpha)$ *is injective for all* e.

Proof. Localization at R° does not kill any element of R (or $\mathbf{F}^e(R) \cong R$) and commutes with $\mathbf{F}^e$. But after localization R becomes a product of fields and α splits. □

Our primary objective in the remainder of this section is to study the case where $N = R$, R is reduced, and $\alpha\colon R \to M$ is injective. In this case $Q = M/\alpha(R)$. Putting (5.8), (5.9), and (5.12) together we have:

(5.13) Theorem. *Let* R *be reduced. An exact sequence* $0 \to R \xrightarrow{\alpha} M \to Q \to 0$ *is a phantom extension if and only if the corresponding element* $\varepsilon \in \mathrm{Ext}^1_R(Q, R)$ *is phantom in the sense described in* (5.8b). *If it is a phantom extension and* R *is weakly F-regular, then the sequence splits, i.e.* $\alpha(R)$ *is a direct summand of* M.

In the sequel we shall use (5.13) to show that a weakly F-regular ring of characteristic p is a direct summand of its module-finite extensions—this is a substantial generalization of the result of [Ho2], which does the case where the ring is regular. Before exploring in this direction, however, we want to show these ideas can be used to prove the existence of big Cohen-Macaulay modules. We first note:

(5.14) Proposition. *Suppose that* $R \xrightarrow{\alpha} M \to Q \to 0$ *is a phantom extension. Then* $\alpha(1)$ *is a basic element in* M; *i.e. it is part of a minimal basis for* M_P *over* R_P *for every prime ideal* P *of* R.

Proof. By (5.7a) the extension remains phantom when we localize, and so it suffices to prove that if (R, m) is local then $\alpha(1) \notin mM$. But if $\alpha(1) \in mM$ then (5.7c) implies that $1 \in m^*$, a contradiction. □

(5.15) Discussion. We now give the details of the proof of the existence of big Cohen-Macaulay modules in characteristic p sketched in the introductory paragraphs for this section. Let (R, m) be a local ring with system of parameters $x_1, \cdots, x_n$. Our objective is to show that there exists an

R-module M_∞, not necessarily finitely generated, such that $x_1, \cdots, x_n$ is a regular sequence on M_∞. Part of the definition of "regular sequence" here is that $(x_1, \cdots, x_n)M_\infty \neq M_\infty$, which is readily seen to be equivalent to the condition $mM_\infty \neq M_\infty$. The idea of the proof is to start with $M_0 = R$, $w_0 = 1$, and construct a sequence of modules M_t with distinguished elements $w_t \in M_t$ together with maps $\mu_t \colon M_t \to M_{t+1}$ (so that the M_t become a direct limit system) such that

(1) $\mu_t(w_t) = w_{t+1}$ for all $t \geq 0$,

(2) for every k, $0 \leq k \leq n-1$, if

$$x_{k+1}m = x_1m_1 + \cdots + x_km_k \tag{†}$$

holds in M_t then for some $T \geq t$, the image of m in M_T is in $(x_1, \cdots, x_k)M_T$, and

(3) $w_t \notin mM_t$ for all t.

If one can construct such a sequence one takes $M = \varinjlim_t M_t$. Condition (2) guarantees that the x_i form a "possibly improper" regular sequence on M, while (3) implies that the element $w_\infty \in M_\infty$ corresponding to the sequence of w_t is not in mM. One easily constructs such a sequence by taking each map μ_t to be a "modification" of M_t in the following sense: One has a relation (†) with m, $m_i \in M_t$, and one takes $M_{t+1} = (M_t \oplus G)/Rh$, where G is the free module on generators $f_1, \cdots, f_k$ and where $h = -m \oplus x_1f_1 \oplus \cdots \oplus x_kf_k$. The map μ_t is the composition of the canonical inclusion of M_t in $M_t \oplus G$ with the canonical quotient surjection, and $w_{t+1} = \mu_t(w_t)$.

After each M_t is constructed, for each k one chooses a finite set of generators for the relations on $x_1, \cdots, x_k, x_{k+1}$ with coefficients in M_t. One "gets rid of" these relations one at a time by successive modifications of the type described above. When one is ready to consider a certain one of these relations one takes its image in the furthest modification M_T yet constructed and then again makes the modification construction exhibited just above. After trivializing all the chosen generating relations coming from M_0, one does the same for those coming from M_1, and so forth. In this way, one obtains a sequence in which (1) and (2) are satisfied. The hardest part is to show that after one has made many modifications, one still has $w_t \notin mM_t$.

The key point from our present perspective is that one can prove that, under mild conditions on R, the map $R \to M_t$ is a phantom extension for all t. It is then immediate from (5.14b) that every $w_t \notin mM_t$, and this is all that is needed. In the proof of the existence of big Cohen-Macaulay modules one is free to replace R by its completion and kill a minimal prime in such a way as to keep the dimension equal to n. Therefore, there is no loss of generality in assuming that R is a complete local domain. Under these conditions (and much more generally) one knows that for part of a system of parameters $x_1, \cdots, x_{k+1}$ one has

$$(x_1^q, \cdots, x_k^q)R :_R x_{k+1}^q R \subseteq ((x_1^q, \cdots, x_k^q)R)^*.$$

If c is a test element then one has

$$c((x_1^q, \cdots, x_k^q)R :_R x_{k+1}^q R) \subseteq (x_1^q, \cdots, x_k^q)R$$

for all q. The existence of big Cohen-Macaulay modules in characteristic p is now immediate from the following easy lemma.

(5.16) Lemma. *Let $R \xrightarrow{\alpha} M$ be a phantom extension and let $x_1, \cdots, x_{k+1}$ be elements of R such that for some $c \in R^\circ$, and all $q \gg 0$, $c((x_1^q, \cdots, x_k^q)R :_R x_{k+1}^q R) \subseteq (x_1^q, \cdots, x_k^q)R$. Suppose that we have a relation (†) $x_{k+1}m = \sum_{i=1}^k x_i m_i$. Let $M' = (M \oplus G)/Rh$, where G is the free module on generators $f_1, \cdots, f_k$, and where $h = -m \oplus x_1 f_1 \oplus \cdots \oplus x_k f_k$, and let $\mu: M \to M'$ be the canonical map. Then $\mu \circ \alpha: R \to M'$ is a phantom extension.*

Proof. Let $w = \alpha(1)$. Let $c' \in R_0$ be such that for all $e \gg 0$ we have $\gamma_e: \mathbf{F}^e(M) \to \mathbf{F}^e(R) = R$ such that $\gamma_e(w^q) = c'$. From (†) we have $x_{k+1}^q m^q = \sum_{i=1}^k x_i^q m_i^q$ and applying γ_e yields $x_{k+1}^q \gamma_e(m^q) \in I^{[q]}$, where $I = (x_1, \cdots, x_k)R$. But then $\gamma_e(m^q) \in I^{[q]} :_R x_{k+1}^q R$, and so $c\gamma_e(m^q) \in I^{[q]}$, say (††) $c\gamma_e(m^q) = \sum_{i=1}^k x_i^q r_i$. We now define $\Gamma_e: \mathbf{F}^e(M') \to R$ by defining it on the numerator $\mathbf{F}^e(M) \oplus Rf_i^q \oplus \cdots \oplus Rf_k^q$ by letting it be $c\gamma_e$ on $\mathbf{F}^e(M)$ and by letting its value on f_i^q be r_i for $1 \le i \le k$. The equation (††) says precisely that the resulting map kills h^q, and so gives a well-defined map $\Gamma_e: \mathbf{F}^e(M') \to R$ (think of $\mathbf{F}^e(M')$ as $(\mathbf{F}^e(M) \oplus Rf_1^q \oplus \cdots \oplus Rf_k^q)/Rh^q$) whose value on $\mu(w)^q$ is cc'. Since c and c' are both independent of e, the existence of the maps Γ_e for $e \gg 0$ shows that $\mu \circ \alpha$ is phantom. □

We next turn our attention to splitting in module-finite extensions and related issues. The following result, which seems rather surprising at first, shows that phantom extensions are abundant in nature.

(5.17) Theorem. *Let R be reduced and S a module-finite overring. Then the inclusion map $R \xrightarrow{\alpha} S$ is a phantom extension. Hence, if R is weakly F-regular, α splits.*

Proof. If we localize at the multiplicative system R° we have that $(R^{\circ})^{-1}R$ is a product of fields, and so $(R^{\circ})^{-1}\alpha$ splits. By restricting the splitting to S and clearing denominators we obtain an R-linear map $\eta: S \to R$ whose value on $1 \in S$ is an element $c \in R^{\circ}$. There is an R-linear map $\theta_e: \mathbf{F}_R^e(S) \to \mathbf{F}_S^e(S) = S$ which sends $r \otimes s$ to rs^q. Then we may take $\gamma_e = \eta \circ \theta_e$, for $\eta(\theta_e(\mathbf{F}^e(\alpha)(1))) = \eta(\theta_e(1 \otimes 1)) = \eta(1) = c$. Since this holds for every e, α is a phantom extension. □

(5.18) Remark. This argument may be less confusing if we adopt the point of view of the discussion in Remark 5.4, which we can do if S is also reduced. From that perspective we are trying to give an $R^{1/q}$-linear map of $R^{1/q} \otimes_R S$ to $R^{1/q}$. Both $R^{1/q}$ and S map R-linearly into $S^{1/q}$, and we obtain an $R^{1/q}$-linear map $R^{1/q} \otimes_R S \to S^{1/q}$. Since the map $R^{1/q} \to S^{1/q}$ is isomorphic with the map $R \to S$, there is an $R^{1/q}$-linear map $\eta^{1/q}: S^{1/q} \to R^{1/q}$ that sends 1 to $c^{1/q}$, and we take γ_e to be the composite.

The idea underlying the proof of (5.17) can also be used to prove for many ring extensions $R \subseteq S$ in characteristic p that $IS \cap R \subseteq I^*$ for every ideal I of S. This is true when R is reduced and S is module-finite over R. More generally, one can weaken these hypotheses to the existence of an R-linear map $\eta: S \to R$ such that $\eta(1) = c \in R^{\circ}$. Let us isolate the following fact, which was established in the proof of (5.17).

(5.19) Proposition. *If S is a module-finite extension ring of a reduced ring R, then there is an R-linear map $\eta: S \to R$ such that $\eta(1) = c \in R^{\circ}$.*

(5.20) Discussion and definitions. In the sequel, we want to consider not only contractions of expanded ideals, but contractions of expanded modules as well. More specifically, we shall be trying to find conditions on a ring homomorphism $R \to S$ such that for every pair of finitely generated R-modules $N \subseteq M$, the inverse image of $N_{S,M} = \operatorname{Im}(S \otimes_R N \to S \otimes_R M)$ (or of its tight closure $N_{S,M}^*$ in $S \otimes_R M$) in M under the obvious map $M \to S \otimes_R M$ sending m to $1 \otimes m$ is contained in N^*. We refer to the inverse image of a submodule of $S \otimes_R M$ in M as its *contraction* to M. Of course, when $M = R$ and $N = I$ is an ideal of R, this simply asks whether the contraction of IS (or $(IS)^*$) to R is contained in I^*.

We want to point out that in this situation it suffices to consider the case where M is a finitely generated free module. In the general case one may map a finitely generated free module G onto M, let H be the inverse image of N in G, and then ask instead whether the inverse image of $H_{S,G}$ or $H_{S,G}^*$ in G is contained in H^*. If $g \in G$ and $\overline{g}$ is its image in M, then it is easy to see that $1 \otimes g$ is in $\operatorname{Im}(S \otimes H \to S \otimes G) = H_{S,G}$

(respectively, in $H^*_{S,G}$) if and only if $1 \otimes \overline{g}$ is in $N_{S,M}$ (respectively, in $N^*_{S,M}$) while g is in H^* if and only if $\overline{g}$ is in N^*. See (3.2). In the case where M is free, we use the less cumbersome notation NS for $N_{S,M}$, by analogy with the case where N is an ideal of R. Moreover, M injects into $S \otimes_R M$ when $R \subseteq S$ and M is free, and we write $NS \cap M$ for the contraction of NS to M.

(5.21) Proposition. *Let $R \to S$ be a homomorphism of Noetherian rings of characteristic p, and let $N \subseteq M$ be finitely generated R-modules. Let R', S' denote R_{red}, S_{red}, respectively. Let $M' = R' \otimes_R M$ and let N' be the image of N in M'.*

(a) *If the contraction of $N'_{S',M'}$ to M' is contained in N'^* then the contraction of $N_{S,M}$ is contained in N^*.*

(b) *The contraction of $(N'_{S',M'})^*$ to M' is contained in N'^* if and only if the contraction of $N^*_{S,M}$ is contained in N^*.*

Proof. Let u be an element of M and u' its image in M'. We know that $u \otimes 1$ is in $N^*_{S,M}$ if and only if $u' \otimes 1$ is in $N'^*_{S_{\mathrm{red}},M'}$ by (3.3b), and that, likewise, u is in N^* if and only if u' is in N'^*. These assertions immediately prove (b). Part (a) then follows at once from the additional remark that if $u \otimes 1$ is in $N_{S,M}$ then u' is in $N'_{S',M'}$. □

By virtue of this result, the questions that we are pursuing can be reduced to the case where the rings are reduced. In the sequel we frequently assume, at least, that R is reduced.

We next observe

(5.22) Theorem. *Suppose that $R \subseteq S$ are Noetherian rings of characteristic p and that there is an R-linear map $\eta: S \to R$ such that $\eta(1) = c \in R^\circ$.*

(a) *For every finitely generated R-module M with $N \subseteq M$, the contraction of $N_{S,M}$ to M is contained in N^*.*

In particular, if I is an ideal of R, then $IS \cap R \subseteq I^*$.

(b) *If, moreover, every element of S° has a multiple in R°, or if S has a weak test element d that has a multiple in R°, then with $N \subseteq M$ as in* (a) *the contraction of $N^*_{S,M}$ to M is contained in N^*.*

In particular, if $I \subseteq R$, then $(IS)^* \cap R \subseteq I^*$.

Proof. Throughout the proof we may assume that M is free. We identify M with R^t. We may then identify $\mathbf{F}^e(S \otimes_R M)$ with S^t. Applying η coordinatewise gives a retraction from $\mathbf{F}^e(S \otimes_R M)$ to $\mathbf{F}^e(M)$ for every e.

(a) Let $u \in NS \cap M$. Then for every q, $u^q \in N^{[q]}S \cap \mathbf{F}^e(M)$. Applying η coordinatewise to both sides yields that $cu^q \in N^{[q]}$.

(b) Observe that if $u \in (NS)^*$ then we can choose d in S° having a multiple in R° such that $du^q \in N^{[q]}S$ for all $q \gg 0$. By replacing d by a suitable multiple in R° we may assume that $d \in R^\circ$. We now apply η coordinatewise to both sides as above to conclude that $cdu^{[q]} \in N^{[q]}$ for all $q \gg 0$, and the result follows. □

(5.23) Corollary. *Let R be a Noetherian ring of characteristic p. If S is a module-finite overring of R, then the contraction of $N^*_{S,M}$ to M is contained in N^* for every pair of finitely generated R-modules $N \subseteq M$. In particular, for every ideal I of R, $(IS)^* \cap R \subseteq I^*$.*

Proof. The hypotheses are preserved if we replace $R \subseteq S$ by the induced inclusion $R_{\mathrm{red}} \subseteq S_{\mathrm{red}}$. Thus, we may assume that the rings are reduced, by Proposition (5.21).

It will suffice to see that every element of S° has a multiple in R°. Note that $A = (R^\circ)^{-1}R \subseteq B = (R^\circ)^{-1}S$ is a module-finite extension, and so $\dim B = \dim A = 0$. The image $s/1$ of any element s of S° in B will be a unit, since it will not be in any minimal prime of B. This is easily seen to imply that s has a multiple in R°. □

(5.24) Remark. Theorem (5.22) is quite close in spirit and proof to the theorem that a module-finite extension of a reduced ring is a phantom extension. We can use it to give an alternate proof of the last statement in (5.17), which, for emphasis, we restate as a theorem in its own right.

(5.25) Theorem. *A weakly F-regular ring R is a direct summand of every module-finite extension S.*

Proof. The result is local on the maximal ideals of R. Since weakly F-regular rings are normal, they are approximately Gorenstein (cf. [HH4], and (8.6); and [Ho4]), and by the results of [Ho4], it suffices to show that every ideal of R is contracted from S. But this is immediate from (5.22) and the fact that every ideal of S is tightly closed. (Alternatively, it suffices to prove that $R \to S$ is *pure*, for which it suffices to show that $M \to M \otimes S$ is injective for every finitely generated R-module M. (Cf. [HR1] and [HR2].) The kernel is the contracted expansion of 0 in M, and so is contained in the expanded contraction of 0, which we know is contained in $0^*_M = 0$.) □

A key point about module-finite extensions is that if $R \to S$ is a module-finite extension and m is any maximal ideal of R, then big ht $mS = \mathrm{ht}\, m$, where the *big height* of an ideal is the supremum of the heights of the minimal primes of I. (Every minimal prime Q of mS lies over m; it is trivial that height $Q \le \mathrm{ht}\, m$. We can obtain equality for at least one choice of Q by applying the going up theorem to a chain of primes of maximum length such that m is its largest element.)

Our next objective is to generalize (5.22) from the case of module-finite extensions to the case of arbitrary homomorphisms that "preserve" a suitable notion of height sufficiently well. As the homomorphism becomes very arbitrary we need to impose some additional conditions on R. The culmination of these results is given in Theorems (5.31) and (5.32) below.

We begin by proving the following two preliminary results.

(5.26) Lemma. *If (A, m) is a complete regular local ring containing a field with $A \subseteq S$ and* big ht $mS = \mathrm{ht}\, m$ $(= \dim A)$, *then the inclusion $A \to S$ splits.*

Proof. By an argument of M. Auslander (see the second paragraph of the proof of Corollary 6.24 of [HH4]), it suffices to show that the map $A \to S$ is pure, and by the technique of [Ho2] it suffices to show for a regular system of parameters $x_1, \cdots, x_n$ in A that for every positive integer t one has $(x_1 \cdots x_n)^t \notin (x_1^{t+1}, \cdots, x_n^{t+1})S$. But if this held in S, it would continue to hold after localization at a minimal prime of $(x_1, \cdots, x_n)S$ whose height is big height $mS = n$. But then the images of the x's in the local ring of S will be a system of parameters, and this would violate the monomial conjecture [Ho2] for the equicharacteristic ring S. $\square$

(5.27) Lemma. *If (R, m) is a complete local domain containing a field, and $R \subseteq S$ with* big ht $mS = \mathrm{ht}\, m$ $(= \dim R)$, *then there is an R-linear map $\eta: S \to R$ such that $\eta(1) \in R^{\circ}$, i.e., $\eta(1) \neq 0$.*

Proof. R is module-finite over a regular local ring $A \subseteq R$. By Lemma 5.26 there is an A-linear splitting $\rho: S \to A$ of the inclusion $A \to S$. This gives an R-linear map $\lambda: S \to \mathrm{Hom}_A(R, A)$ defined by $\lambda(s)(r) = \rho(rs)$ whose value on 1 is the map $\rho|_R$, which is nonzero, since $\rho(1) = 1$. Since $\mathrm{Hom}_A(R, A)$ is a rank one torsion-free module over R, it has an embedding $\mu: \mathrm{Hom}_A(R, A) \to R$, and we may take $\eta = \mu \circ \lambda$. $\square$

In the reduced case we need to consider preservation of height modulo every minimal prime. Note first that when R is Noetherian and $R \subseteq S$, every minimal prime of R (and every associated prime) is contracted from S: the point is that if $P = \mathrm{Ann}_R u$ with $u \in R$, then $P = \mathrm{Ann}_S u \cap R$.

(5.28) Lemma. *If (R, m) is a reduced complete local ring containing a field, and we have an injection homomorphism $R \subseteq S$ with* big ht $m(S/PS) = \mathrm{ht}\, m/P$ $(= \dim R/P)$ *for every minimal prime P of R, then there is an R-linear map $\eta: S \to R$ such that $\eta(1) \in R^{\circ}$.*

Proof. Let $P_1, \cdots, P_h$ be the minimal primes of R. For every P_j Lemma (5.27) guarantees a map $S/P_jS \to R/P_j$ whose value on 1 is nonzero, and hence a map $S \to R/P_j$ with the same property. Together, these give a map $\lambda: S \to \prod_j R/P_j$ such that the value on 1 is nonzero in each component. The obvious map $R \to \prod_j R/P_j$ becomes an isomorphism if we localize at R°: the inverse, restricted to $\prod_i R/P_i$, gives an injection $\prod_j R/P_j \to (R^\circ)^{-1}R$, and if we multiply by a suitable element of R° to clear denominators, we obtain an injection $\mu: \prod_j R/P_j \to R$ which becomes an isomorphism when we localize at R°. The elements in the product all of whose components are nonzero map to R° under the homomorphism μ: thus, we may take $\eta = \mu \circ \lambda$. □

(5.29) Definition. A ring homomorphism $R \to S$ is called *formally height preserving* if for every maximal ideal m of R and every minimal prime P of $B = (R_m)^\frown$, if T denotes the mS-adic completion of S then $\operatorname{big\,ht} m(T/PT) \geq \operatorname{ht} m(B/P) = \dim B/P$. (We may view B as simply the m-adic completion of R, so that B does, in fact, map to T.) The following result contains some basic facts about formally height preserving homomorphisms.

(5.30) Proposition. *Let $R \to S$ be a homomorphism of Noetherian rings.*

(a) *Fix a maximal ideal m of R and a minimal prime Q of the completion B of R_m. Let T be the mS-adic completion of S. Then* $\operatorname{big\,ht} m(T/QT) \geq \dim B/Q$ *if and only if there exists a minimal prime ideal $\mathfrak{m}$ of mS in S such that, with $C = (S_{\mathfrak{m}})^\frown$, $\dim C/QC \geq \dim B/Q$, in which case equality holds.*

(b) *$R \to S$ is formally height preserving if and only if $R_{\text{red}} \to S_{\text{red}}$ is formally height preserving.*

(c) *If $R \to S$ is a module-finite extension then it is formally height preserving.*

(d) *If $R \to S$ is faithfully flat then it is formally height preserving.*

(e) *If $R \to S$ is formally height preserving then for every local ring B that is the completion of R at a maximal ideal m if T denotes the (mS-adic) completion of S then there is B_{red}-linear map from $T_{\text{red}} \to B_{\text{red}}$ such that the value on 1 is in $(B_{\text{red}})^\circ$.*

Proof. (a) Fix a minimal prime Q of B. Then $\operatorname{big\,ht} m(T/QT) \geq \dim B/Q$ if and only if there exists a minimal prime $\mathfrak{q}$ of mT in T such that $\operatorname{ht} \mathfrak{q}/QT \geq \dim B/Q$. Since $T/mT \cong S/mS$, every minimal prime $\mathfrak{q}$ of mT corresponds to a minimal prime $\mathfrak{m}$ of mS. Next observe that the completion of $T_{\mathfrak{q}}$ is isomorphic with the completion of $S_{\mathfrak{m}}$: we have that $\mathfrak{q}$ contracts to $\mathfrak{m}$, which yields an induced map $S_{\mathfrak{m}} \to T_{\mathfrak{q}}$ and, hence, a map of the completions. On the other hand, the map $S \to S_{\mathfrak{m}}$ yields an induced map of the mS-adic completion T of S

to $(S_{\mathfrak{m}})^{\wedge}$. The maximal ideal of this last completion lies over $\mathfrak{q}$ in T, and this yields induced maps $T_{\mathfrak{q}} \to (S_{\mathfrak{m}})^{\wedge}$ and, finally, $(T_{\mathfrak{q}})^{\wedge} \to (S_{\mathfrak{m}})^{\wedge}$. It is easy to see that the two maps $(S_{\mathfrak{m}})^{\wedge} \to (T_{\mathfrak{q}})^{\wedge}$ and $(T_{\mathfrak{q}})^{\wedge} \to (S_{\mathfrak{m}})^{\wedge}$ that we have constructed are mutual inverses. Now, big $\operatorname{ht} mT/QT \geq \operatorname{ht} \mathfrak{q}/QT = \dim T_{\mathfrak{q}}/QT_{\mathfrak{q}} = \dim (T_{\mathfrak{q}})^{\wedge}/Q(T_{\mathfrak{q}})^{\wedge} = \dim (S_{\mathfrak{m}})^{\wedge}/Q(S_{\mathfrak{m}})^{\wedge}$, with equality for at least one choice of $\mathfrak{q}$ and corresponding $\mathfrak{m}$. It follows that big $\operatorname{ht} mT/QT \geq \dim B/Q$ if and only if there exists a minimal prime $\mathfrak{m}$ of mS in S such that, with $C = (S_{\mathfrak{m}})^{\wedge}$, $\dim C/QC \geq \dim B/Q$. Since the maximal ideal $\mathfrak{m}C/QC$ of C/QC is a minimal prime of the expansion mC/QC of the maximal ideal mB/Q of B/Q, this inequality holds if and only if $\dim C/QC = \dim B/Q$.

(b) This is an easy consequence of the fact that killing the nilpotents in a ring does not affect the dimension, and of the fact that big $\operatorname{ht} J =$ big $\operatorname{ht} JT_{\mathrm{red}}$ for any ideal J of any Noetherian ring T.

(c) Since S is module-finite over R, the mS-adic completion of S is the same as $B \otimes_R S$, where B is the m-adic completion of R, and since B is flat over R, applying $B \otimes_R$ preserves the injectivity of $R \to S$. Thus, we need only study the case where R is a complete local ring and S is module-finite over R. Then every minimal prime of R is contracted from S, and so when we replace R, S by R/P, S/PS with P a minimal prime of R we still have a module-finite extension. But it has already been observed that, for a module-finite extension S of a local ring (R, m, K), big $\operatorname{ht} mS = \operatorname{ht} m$, which completes the proof of (a).

(d) Choose any minimal prime $\mathfrak{m}$ of mS. Then, with notation as in part (a), $R_m \to S_{\mathfrak{m}}$ is faithfully flat and so the induced map of completions $B \to C$ is faithfully flat. Likewise, $B/PB \to C/PC$ is faithfully flat, and so $\dim C/PC \geq \dim B/PB$.

(e) This is immediate from the definition of formally height preserving, the fact that killing nilpotents affects neither dimension nor big height, and Lemma (5.28). $\square$

We are now ready to prove one of our main results.

(5.31) Theorem. *Let $R \to S$ be a formally height preserving homomorphism of Noetherian rings of characteristic p, and suppose also that R has a completely stable weak test element.*

Then for every inclusion of finitely generated R-modules $N \subseteq M$, the contraction of $N_{S,M}$ to M is contained in N^*. In particular, for every ideal I of R, $IS \cap R \subseteq I^*$.

Suppose, in addition, that either

(i) *S is essentially of finite type over R, or, more generally, that for* *every maximal ideal m of R and every prime ideal $\mathfrak{m}$ of S minimal over mS, the fields $S_{\mathfrak{m}}/\mathfrak{m}S_{\mathfrak{m}}$ is finitely generated as a field over R/m or*

(ii) *the residue fields of R modulo its maximal ideals are perfect.*

Then for every inclusion of finitely generated R-modules $N \subseteq M$ the contraction of $N^*_{S,M}$ to M is contained in N^*. In particular, for every ideal I of R, $(IS)^* \cap R \subseteq I^*$.

Proof. Suppose that we have an element of the contraction of $N_{S,M}$ (or $N^*_{S,M}$) to M that is not in N^*. Then the situation is preserved when we replace R by its completion B at a suitable maximal ideal m (because R has a completely stable test element) and S by its mS-adic completion T. We can then pass to B_{red} and T_{red} without affecting the condition on the heights. The result for the case where the element is in the contraction of $N_{S,M}$ is now immediate from (5.30e) and (5.22).

Now assume that either (i) or (ii) holds and that we have an element in the contraction of $N^*_{S,M}$ to M that is not in N^*. As we have already observed in the first paragraph, we may assume without loss of generality that (R, m, K) is a complete local ring. We may also replace S by its localized completion at the union of the minimal primes of $\mathfrak{m}S$, which is the same as the product of its localized completions at the minimal primes of $\mathfrak{m}S$. Note that the Jacobson radical of the new S will be the radical of mS. Thus, we may assume that S is complete semilocal and that either (i) every residue class field at a maximal ideal is finitely generated as a field over K or (ii) K is perfect.

Now, the fact that an element is outside the tight closure of N in M will be preserved when we kill a suitable minimal prime P of R, by (3.3d). We can then replace S by $S_{\mathfrak{m}}/PS_{\mathfrak{m}}$ for a suitable maximal ideal $\mathfrak{m}$ (chosen so that $\dim S_{\mathfrak{m}}/PS_{\mathfrak{m}} = \dim R/P$). Note that the fact that a certain element is in a tight closure in an S-module is preserved both by localizing at $\mathfrak{m}$ and then by killing $PS_{\mathfrak{m}}$ in the complete local ring $S_{\mathfrak{m}}$ (by Theorem 6.24 of [HH9]).

In this way we obtain a counterexample in which (R, m, K) is a complete local domain, $(S, \mathfrak{m}, L)$ is a complete local extension ring, $\dim S = \dim R$, $\mathfrak{m}$ is minimal over mS (so that any system of parameters for R is a system of parameters for S) and either (i) L is finitely generated as a field over K or (ii) K is perfect. Choose a minimal prime $\mathfrak{n}$ of S such that $\dim S/\mathfrak{n} = \dim S$ $(= \dim R)$. Then $\mathfrak{n}$ cannot meet R: if it did, we could choose fewer than $\dim R$ elements in m whose images in $R/\mathfrak{n}$ generate an $(m/\mathfrak{n})$-primary ideal, and these same elements would then generate an $(\mathfrak{m}/\mathfrak{n}S)$-primary ideal in $S/\mathfrak{n}S$ (since mS is

$\mathfrak{m}$-primary), a contradiction, since $\dim S/\mathfrak{n}S = \dim R$. It follows that we may replace S by $S/\mathfrak{n}S$ without affecting any of our hypothesis. Thus, we henceforth assume that $R \to S$ is an injection of complete local domains.

Let S' be the normalization of S, which is a complete local ring module-finite over S. Then we may replace S by S' as well, and we henceforth assume that S is normal. Now let R' be the normalization of R, which may be identified with a subring of S. Suppose we know the theorem for the map $R' \to S$, and apply it to the pair of modules $M' = R' \otimes_R M$ and $N' = \mathrm{Im}(R' \otimes_R N \to R' \otimes_R M) = N_{R',M}$. Then we may think of $N_{S,M}$ as $(N_{R',M})_{S,M'} = (N')_{S,M'}$ and so we find that the contraction of $N^*_{S,M} = ((N')_{S,M'})^*$ to M' is contained in N'^*. But the contraction of N'^* to M is contained in N^* since we already know the module-finite case. Thus, we may replace R by R' and we henceforth assume that R is normal as well.

Choose a coefficient field $K \subseteq R$ for R.

In case (i) let $s_1, \cdots, s_h$ denote elements of S whose images in L form a transcendence basis for L over K. Map the polynomial ring $R[y] = R[y_1, \cdots, y_h]$ R-linearly to S by sending y_i to s_i for $1 \le i \le h$. Any polynomial outside $mR[y]$ maps to a unit of S. Thus, there is an R-homomorphism of the localization $R(y)$ of $R[y]$ at $mR[y]$ to S that sends the y_i to the s_i. Let $R\langle y\rangle$ denote the completion of $R(y)$ with respect to the maximal ideal. Then $R\langle y\rangle$ also maps to S. Since the maximal ideal of $R\langle y\rangle$ expands to an ideal primary to the maximal ideal of S and since the extension of residue fields $K(y) \subseteq L$ will also be finite algebraic, it follows from the completeness that S is module-finite over the image of $R\langle y\rangle$.

Now $\dim S = \dim R = \dim R(y) = \dim R\langle y\rangle$ here. Since R and, hence, $R(y)$ are normal local (and excellent), we have that $R\langle y\rangle$ is a domain, and it follows that $R\langle y\rangle \to S$ is injective as well. Since $R\langle y\rangle \to S$ is module-finite, we may apply Corollary (5.23) to conclude that every element of the contraction of $N^*_{S,M}$ to M is in $N^*_{R\langle y\rangle,M}$, and hence in the contraction of $N^*_{R\langle y\rangle,M}$, and hence in the contraction of $N^*_{R\langle y\rangle,M}$ to M. Thus, we have reduced to studying the case where $S = R\langle y\rangle$. By Theorem (7.36) of [HH9], a completely stable test element for R (these exist, by Theorem (6.2) of [HH9]) will also be a test element for $R\langle y\rangle$. The result when $S = R\langle y\rangle$ is now immediate from this remark, Theorem (5.22b), Proposition (5.30d) (which guarantees that $R \to R\langle y\rangle$ is formally height preserving), and Lemma (5.28).

It remains to consider case (ii), where K is perfect. Then R is module-finite over $A = K[[x_1, \cdots, x_d]]$ where $x_1, \cdots, x_d$ is a system of parameters for R. Since the coefficient field K for R is perfect, its image in S can be expanded to a coefficient field L for S. The x's can also be viewed as a system of parameters for S. Thus, S is module-finite over $B = L[[x_1, \cdots, x_d]]$. The ring $R_L = B \otimes_A R$ is faithfully flat over R (since B is faithfully flat over A) and module-finite over B (since R is module-finite over A). We have A-homomorphisms $R \to S$ and $B \to S$, yielding an A-homomorphism $R_L \to S$, and since S is module finite over B, it is module-finite over the image, S_0, of $R_L \to S$. Since R_L is complete and semilocal, S_0 is complete and semilocal. But since S is a domain, so is S_0. Thus, S_0 must be local. It follows that S_0 must be the quotient of R_L be a minimal prime $\mathfrak{p}$. Since S is module-finite over S_0, we can replace S by S_0. We change notation, and henceforth assume that $S = R_L/\mathfrak{p}$, where $\mathfrak{p}$ is a minimal prime of R_L disjoint from $R - \{0\}$ such that $\dim R_L/\mathfrak{p} = \dim R$.

To complete the argument we want to apply Theorem (5.22b) in this situation. Lemma (5.27) yields one part of the needed hypothesis. To obtain the other part, it will suffice to show that there is an element $c \in R - \{0\}$ which is a test element for S. By Theorem (6.2) of [HH9], it suffices to show that there exists an element $c \in R - \{0\}$ such that S_c is regular, and this will follow if $(R - \{0\})^{-1}S = F \otimes_R S$ is regular, where F denotes the fraction field of R. Since $S = R_L/\mathfrak{p}$, where $\mathfrak{p}$ is a minimal prime of R_L, it will suffice to show that $F \otimes_R R_L$ is regular: $F \otimes_R S$ will then be one of the factors when $F \otimes_R R_L$ is written as a product of regular domains.

Now, $F \otimes_R R_L$ is the same as $F \otimes_R (R \otimes_A B) \cong F \otimes_A B$, where from here on we simply regard F as some finite field extension of the fraction field G of A. For the purpose of seeing that this ring is regular, we may replace F by a larger field. We may therefore view F as a purely inseparable extension of G followed by a separable extension. But the separable part is harmless, while the purely inseparable part is contained in the fraction field of $K[[x_1^{1/q}, \cdots, x_d^{1/q}]] = A^{1/q}$ for any sufficiently large $q = p^e$, since K is perfect. We have therefore reduced to the case where F is the fraction field of $A^{1/q}$, and then $F \otimes_A B$ is a localization of $A^{1/q} \otimes_A B \cong L[[x_1^{1/q}, \cdots, x_d^{1/q}]]$, which is regular. □

We next observe:

(5.32) Theorem. *Let R be a weakly F-regular ring of characteristic p and suppose either that*

(a) *R is complete local or*

(b) *R has a completely stable test element.*
Suppose also that $R \subseteq S$ Noetherian and that big ht $mS = $ ht m *for every maximal ideal m of R. Then $IS \cap R = I$ for every ideal I of R, and so R is pure in S. In case* (a) *R is a direct summand of S.*

Proof. As mentioned in the proof of (5.26), by an argument of M. Auslander (again, see the second paragraph of the proof of Corollary 6.24 of [HH4]), a map from a complete local ring R is split if and only if it is pure, and, by the results of [Ho4] on approximately Gorenstein rings, a map from a normal ring R is pure if every ideal is contracted. Thus, all of the results will follow if we can show in cases (a) and (b) that $IS \cap R = I$ for all ideals I of R. Case (a) is now immediate from (5.28) coupled with (5.22).

We focus on case (b). To show that $IS \cap R = I$ for every ideal I of R it suffices to consider the case where I is primary to a maximal ideal m of R, since, in a Noetherian ring, every ideal is an intersection of ideals primary to maximal ideals. Suppose that I is primary to m, where ht $m = n$, and that Q is a minimal prime in S of mS of height n. It will suffice to consider the map $R_m \to S_Q$, and we shall be able to conclude that the induced map remains injective when both rings are completed provided that we can show that the completion of R_m is a domain. This follows from the fact that R_m and its completion have a common test element: the tight closure of an ideal primary to m in R_m then corresponds to the tight closure of its expansion to $\widehat{R}_m$ (under the obvious bijective correspondence between $m\widehat{R}_m$-primary ideals of $\widehat{R}_m$ and mR_m-primary ideals of R_m given by contraction and expansion) by [HH4, (5.1c)], and it follows that every $m\widehat{R}_m$-primary ideal and, hence, every ideal of $\widehat{R}_m$ are tightly closed. Thus, $\widehat{R}_m$ is a weakly F-regular ring and, consequently, a domain. It follows from part (a) that $I\widehat{R}_m$ is contracted from $\widehat{S}_Q$ for every m-primary ideal I of R. We then have that I is contracted from R_m, that IR_m is contracted from $\widehat{R}_m$, and that $I\widehat{R}_m$ is contracted from $\widehat{S}_Q$. It follows that I is contracted from $\widehat{S}_Q$ and, hence, from S. □

(5.33) Remarks and questions. We know that for module-finite extensions S of a Noetherian ring R of characteristic p we have $IS \cap R \subseteq I^*$. It is natural to ask whether, under mild conditions on R, every element of I^* must be in IS in some module-finite extension S of R. See [Sm1–2] for the case of parameter ideals. We do not know the answer in general. It is also natural to ask for a weaker result: under mild conditions on R, if $I \subseteq R$, can every element of I^* be forced into IS in an extension algebra

S for which there exists an R-linear map $\eta: S \to R$ such that $\eta(1) \in R^\circ$. See [Ho9–10].

In §6 we shall show that, under mild conditions on a Gorenstein ring R (e.g., locally excellent suffices), R is F-regular iff it is a direct summand of every module-finite extension. The main point is that there is a technique for showing that under certain special conditions (roughly, that I be generated by an R-sequence consisting of test elements), elements of I^* can be forced into IS for some module-finite extension algebra R. This technique is based on results from [HH7], which were used there to show that the integral closure of an excellent local domain of characteristic p in an algebraic closure of its fraction field is a big Cohen-Macaulay algebra over the local domain. It is noteworthy that the main result of [HH7] and the techniques used there originated in studying the problem of forcing elements of I^* into I.

In [HH7] it is also shown that, under mild conditions on a ring R of characteristic p, if R is a direct summand of every module-finite extension algebra then R is Cohen-Macaulay. Note that nothing like this is true in characteristic zero: if R contains the rationals and is normal, a trace argument shows that it is a direct summand of every module-finite extension. Thus, while the condition that R be a direct summand of every module-finite extension is very strong in characteristic p, implying the Cohen-Macaulay property as well as normality and perhaps even F-regularity, it is merely equivalent to normality in characteristic 0.

(5.34) Discussion. In Definition 5.2 we required that M be a finitely generated R-module. In this discussion we relax that condition, and assume instead that M is a finitely generated S-module over a Noetherian R-algebra S. The definition still gives a notion of phantom extension in this case. We are interested for the moment in discussing phantom extensions which are algebra maps $R \to S$. A key point is that for an element $u \in R$ and an ideal $I \subseteq R$, if $u \in IS \cap R$ for some R-algebra S such that $R \to S$ is phantom, then $u \in I^*$: this follows, in essence, from (5.7c), for the proof does not use the fact that M is finitely generated as an R-module. This raises the question whether every element of I^* can be forced into IS for some algebra extension $R \to S$ which is phantom.

(5.35) Remarks and questions. In a different direction, we ask whether, if R is a domain of characteristic p (perhaps satisfying some additional "good" conditions, e.g. that $R^{1/p}$ be module-finite over R) and d is an arbitrary fixed nonzero element of R, there must exist an integer $q = p^e$ such that the map $R \to R^{1/q}$ sending 1 to $d^{1/q}$ is a phantom extension. An affirmative answer in the case where $R^{1/p}$ is module-finite over R

would show that weak F-regularity implies strong F-regularity when $R^{1/p}$ is module-finite over R (see [HH3] and §5 of [HH9] for treatments of the notion of *strong* F-*regularity*). This is true when R is Gorenstein but is an open question in general.

A natural way to attack the question raised in the preceding paragraph is to first assume that R is complete local and represent it as a finitely generated module over a regular local subring A such that the given element $d \in A$. To study the easiest case, we assume as well that the fraction field of R is separable over the fraction field of A. It is easy to see that the map $A \to A^{1/q'}$ sending 1 to $d^{1/q'}$ splits for sufficiently large q': fix one such value of q'. If we tensor with R over A we get a split R-linear map $R \to R \otimes_A A^{1/q}$, and by virtue of the separability assumption we may identify $S = R \otimes_A A^{1/q}$ with its image $R[A^{1/q}]$ in $T = R^{1/q}$. This leads to the following question:

(5.36) If $R \subseteq S \subseteq T$ are module-finite extension domains of R and $s \in S$ is an element such that the map $R \to S$ sending 1 to s splits, is the map $R \to T$ sending 1 to s a phantom extension?

An affirmative answer would show, in very good cases, that the map sending $1 \in R$ to $d^{1/q'}$ in $R^{1/q'}$ is phantom for sufficiently large q'. However, we do not know whether (5.36) is true, and the naive intuition which suggests that it ought to be true is flawed. This intuition is based on what happens in the case where maps are split: we are composing the map $R \to S$ which sends 1 to s with the inclusion $S \to T$, which is a phantom extension over S. If it were actually split over S, it would be split over R and the composition would be split over R. Instead, we know only that it is a phantom extension over S. If it were phantom over R we could apply (5.7f) and we would be done. However, as we show in the example below, the fact that $S \to T$ is phantom over S does not imply that it is phantom over R, and so there is no apparent reason for (5.36) to have an affirmative answer.

(5.37) Example. Let T be a complete Cohen-Macaulay domain, let S be a subring of T over which it is module-finite such that S is not Cohen-Macaulay, and represent S as a finitely generated module over a complete regular local ring R contained in S. (For definiteness, we could take $T = K[[x, y]]$, $S = K[[x^4, x^3y, xy^3, y^4]]$, and $R = K[[x^4, y^4]]$.) Then $S \to T$ is phantom over S, since it is module-finite, but it is not phantom over R. If it were, then since $F: R \to R$ is flat, $\mathbf{F}^e_R(S) \to \mathbf{F}^e_R(T)$ is injective for all e, and it follows from Proposition (5.8) that the element in $\mathrm{Ext}^1_R(T/S, S)$ corresponding to the exact sequence of R-modules $0 \to$

$S \to T \to T/S \to 0$ is in the tight closure of 0 in a certain module containing $\mathrm{Ext}^1_R(T/S, S)$. But, since R is regular, this means that the element is zero, i.e., that $S \to T$ splits over R. That is impossible, since S has smaller depth than T as an R-module.

6. Forcing elements into expanded ideals in integral extensions and a characterization of F-regular Gorenstein rings

(6.1) Discussion. One of our main objectives in this section is to prove that under a mild condition on a Gorenstein ring R of characteristic p (that R contain a completely stable test element and have smooth formal fibers), R is F-regular iff for every module-finite extension S of R, R is a direct summand of S. See Theorem (6.7). Note that by (5.17), a weakly F-regular ring is a direct summand of every module-finite extension. Thus, what remains to be done is to show that if a Gorenstein ring is a direct summand of every module-finite extension, then it is F-regular.

The key point is to show that, under special conditions, an element in the tight closure of an ideal generated by parameters can be forced into the expansion of the ideal in a suitable module-finite extension. Then, because the ring is a direct summand of every module-finite extension, the element must have been in the ideal originally. The equations used to define the extensions we used are reminiscent of those defining the standard Artin-Schreier extensions of fields. However, in degenerate cases, they may yield, for example, purely inseparable field extensions.

We note that there cannot be a corresponding theory in characteristic 0, since a normal ring containing the rationals is a direct summand of every module-finite extension, while the ring need not even be Cohen-Macaulay much less F-regular. We do not know whether a ring of characteristic p that is a direct summand of every module-finite extension must be F-regular without the Gorenstein hypothesis. However, it is shown in [HH7] that, under mild conditions on the ring, such a ring must be Cohen-Macaulay. Closely related ideas are used in [HH7] to show that if one has an excellent local domain R then the integral closure of R in an algebraic closure of its fraction field is a big Cohen-Macaulay algebra for R.

In the latter parts of this section, all given rings are Noetherian, of characteristic p. However, we do not impose this restriction generally until (6.6).

A critical tool in all of this is the following:

(6.2) Theorem. *Let R be a Noetherian domain of characteristic p and let $x_1, \cdots, x_k$ be an R-sequence consisting of test elements. Let $I = (x_1, \cdots, x_k)R$. Then there exists a module-finite extension domain S of R such that $I^* \subseteq IS \cap R$.*

Before proving this result, which is contained in Theorem (6.6) below, we need some preliminary facts. One of the crucial results is Theorem (6.3) below, which generalizes a result of [Ma]. Theorem (6.3) is Theorem (2.2) of [HH7].

(6.3) Theorem (equational lemma). *Let R be a ring of characteristic p and let $x_1, \cdots, x_k, z_1, \cdots, z_s \in R$. Let $x = x_1 \cdots x_k$ denote the product of the x_i, and assume that none of the x_i is a zerodivisor. Suppose that for every i, $1 \le i \le s$, we have elements r_{ih} and $r'_{ij} \in R$ such that*

$$(*_i) \qquad z_i^p = \sum_{h=1}^{k} r_{ih} x_h^p + x^{p-1} \sum_{j=1}^{s} r'_{ij} z_j .$$

Then there is a module-finite extension ring S of R such that $(z_1, \cdots, z_s) S \subseteq (x_1, \cdots, x_k)S$. If R is a domain, then S can be chosen to be a domain.

If R is graded by a semigroup $H \subseteq \mathbb{Q}^r$ and the elements $x_1, \cdots, x_k$, $z_1, \cdots, z_s$ are homogeneous, then the ring S may be chosen so that it is graded by $H - H$, and every nonnilpotent homogeneous element of S will have a degree that has a positive multiple in H. □

(6.4) Remark. (a) We can rephrase this as follows. Let R be a ring of characteristic p, let $I = (x_1, \cdots, x_k)R$, where none of the x_i is a zerodivisor in R, and let $x = x_1 \cdots x_k$. Let J be a finitely generated ideal of R (corresponding to the ideal generated by the z's) such that $J^{[p]} \subseteq I^{[p]} + x^{p-1}J$. Then there exists a module-finite extension ring S of R such that $JS \subseteq IS$, and S can be taken to be a domain if R is a domain.

(b) When, moreover, $x_1, \cdots, x_k$ is an R-sequence, then $I^{[p]} : I = I^{[p]} + (x_1 \cdots x_k)^{p-1}R$. If we let $J = (z_1, \cdots, z_s)R$, we see that, since $I^{[p]} + (I^{[p]} : I)J = I^{[p]} + (x_1 \cdots x_k)^{p-1}J$, Theorem (6.3) for the domain case can be restated as follows:

(6.3°) Theorem. *If R is a domain of characteristic p, I is an ideal generated by an R-sequence $x_1, \cdots, x_k$, and J is a finitely generated ideal such that $J^{[p]} \subseteq I^{[p]} + (I^{[p]} : I)J$, then $J \subseteq IR \cap S$ for some module-finite extension domain S of R.*

(6.5) Remark. I. Aberbach, in his thesis [Ab1], showed that this statement remains true without the hypothesis that I be generated by an

R-sequence: the argument is much more subtle, however. The statement is also true, without the hypothesis that I be generated by an R-sequence, if one knows that $J^{[q]} \subseteq I^{[q]} + (I^{[q]} : I)J$ for some power q of p. The obvious possibility of trying to define some sort of closure for I in terms of conditions of this type is obstructed by the fact that J may satisfy such a condition for a subideal I_0 of I without satisfying such a condition for I itself: the problem is that $I^{[q]} : I$ is not a monotone function of I (consider the chain $(0) \subseteq (x) \subseteq (x, y) \subseteq R = K[x, y]$). While various ad hoc measures to get around this difficulty suggest themselves, we have not yet found one that seems natural or optimal.

In any case, it is certainly important to understand precisely what kinds of systems of equations can be used to force elements of R into $IS \cap R$ for module-finite extension domain S of R when R is a normal domain of characteristic p. Such understanding might enable one to show that every element of I^* is in $IS \cap R$ for some such S, and then one could show that tight closure commutes with localization. A related problem is to find equational conditions that force elements of a ring R to be in I^*. In a way, the definition of tight closure is equational, but there are infinitely many equations (one for every large q). A characterization in terms of finitely many equations would, again, very likely enable one to prove that tight closure commutes with localization. The next result moves in this direction, but needs a stringent restriction on I.

(6.6) Theorem. *Let $x_1, \cdots, x_k$ be a regular sequence in a Noetherian domain R of characteristic p such that every x_i is a test element. Let $I = (x_1, \cdots, x_k)R$ and $J = I^*$. Then $J^{[p]} \subseteq I^{[p]} + (I^{[p]} : I)J$, and so there is a module-finite extension domain $S \supseteq R$ such that $I^* \subseteq IS \cap R$.*

Proof. Since every x_i is a test element, $x_i J^{[p]} \subseteq I^{[p]}$, and so $J^{[p]} \subseteq I^{[p]} : I = I^{[p]} + x^{p-1}R$, where $x = x_1 \cdots x_k$. Suppose we have $j \in J^{[p]}$ and we write $j = i + x^{p-1}r$, where $i \in I^{[p]}$ and $r \in R$. To complete the proof, it will suffice to show that $r \in J$. Since $j - i \in J^{[p]}$, we have that $r \in J^{[p]} : Rx^{p-1}$. Since $J = I^*$, $J^{[p]} \subseteq (I^{[p]})^*$ (since $cj^q \in I^{[q]}$ for $q \gg 0$ implies $c^p(j^p)^q \in I^{[p][q]}$ for $q \gg 0$), and so rx^{p-1} is in $(I^{[p]})^*$. Let $I^{[p]} = H$ and $x^{p-1} = y$. Since $ry \in H^*$, for some $c \in R^\circ$ and for all $q \gg 0$, $c(ry)^q \in H^{[q]}$, which implies that $cr^q \in H^{[q]} : y^q R = (H : yR)^{[q]}$ (since we are dealing with monomials in an R-sequence), and so $r \in (H : y)^* = ((x_1, \cdots, x_k)R)^* = J$. This completes the proof that $J^{[p]} \subseteq I^{[p]} + x^{p-1}J$, and the last statement is now immediate from Theorem 6.3°. □

(6.7) Theorem. *Let R be a Gorenstein domain of characteristic p that is a direct summand of every module-finite extension ring, and suppose that for every local ring A of R the map $A \to \widehat{A}$ has regular fibers, which holds, in particular, if R is locally excellent. Then R is F-regular.*

Thus, a locally excellent Gorenstein ring is F-regular if and only if it is a direct summand of every module-finite extension.

Proof. First note that the splitting hypothesis passes to localizations of R: if W is a multiplicative system and T is a module-finite extension of $W^{-1}R$, then we can choose finitely many $W^{-1}R$-module generators of T, and after multiplying each by an element of W we may assume that each is integral over R. These elements then generate a module-finite extension algebra T_0 of R contained in T such that $W^{-1}T_0 = T$. Since $R \to T_0$ splits, so does $W^{-1}R \to W^{-1}T_0 = T$.

Thus, if R is not F-regular we may choose a prime P minimal in $\operatorname{Spec} R$ such that R_P is not F-regular. Henceforth, we replace R by R_P and assume that (R, m, K) is local. If $x \in m$ then R_x is F-regular and Gorenstein (it suffices to see that each of its local rings is weakly F-regular, and this follows from the minimality of P). It follows from Lemma (3.7) that there exists an m-primary ideal $\mathscr{I}$ whose nonzero elements are all test elements for R, and so there is a system of parameters $x_1, \cdots, x_n$ for R such that every x_i is a test element. Since R is Gorenstein, by Proposition (3.8(f)) to prove that R is F-regular it suffices to prove that the ideal $I = (x_1, \cdots, x_n)R$ is tightly closed. Let u be an element of I^*. By Theorem (6.6) we can choose a module-finite extension domain $S \supseteq R$ such that $u \in IS$. Applying an R-module retraction $\rho\colon S \to R$, we find that $u \in I$. Thus, $I = I^*$. □

7. Criteria for F-rationality and F-regularity

In this section we give some criteria for F-rationality. In the Gorenstein case these are, of course, criteria for F-regularity. In particular, we develop criteria in Theorems (7.7), (7.12) and Corollary (7.13) that suffice for most of the applications that we have in mind. In the case of isolated singularities the result of (7.13) can be deduced from the main results of [FeW]. The arguments of [FeW] can be adapted to the generality needed here: in a number of places in [FeW] where it is assumed that one has an isolated singularity, all that is actually needed to make the proofs work is that every element in $R^\circ \cap m$, where (R, m, K) is the local ring of the point, has a power that is a test element. This is known in a number of situations where the singularity need not be isolated: see (7.9), (7.10), and (7.11).

However, we have given a self-contained proof of Theorem (7.12) and Corollary (7.13) using instead Theorem (7.7) below, whose demonstration is based on an idea very similar to the underlying idea of the proof of Theorem (6.6).

We then apply our criterion to prove that Grassmannians are F-regular in characteristic p, as are the rings defined by the vanishing of the minors of fixed size of a matrix of indeterminates. More general results of this kind are obtained in [Gla].

We also treat two examples of Watanabe [W2] and [W3] from the perspective developed in this section: these are F-rational surfaces that are not weakly F-regular (one is F-pure and the other is not).

(7.1) Definitions. If I is an ideal in a ring of characteristic p, we denote by I^F the *Frobenius closure* of I in R (the set of all elements $g \in R$ such that $g^q \in I^{[q]}$ for some $q = p^e$). When $I = I^F$ we say that I is *contracted with respect to Frobenius.* If $F: R \to R$ is *pure,* i.e., if R is F-*pure* in the terminology of [HR2], then every ideal of R is contracted with respect to Frobenius. (The converse is also true under mild conditions on R.)

In (7.2) and (7.3) we do not assume that the ring has characteristic p except where specified.

(7.2) Definition. Let R denote a finitely generated $\mathbb{N}$-graded K-algebra of Krull dimension d, where K is a field and $R_0 = K$. Let m be the homogeneous maximal ideal of R. Recall that the a-invariant, $\mathbf{a}(R)$, is the largest integer a such that $[H^d_m(R)]_a \neq 0$, where $[\,]_a$ indicates the ath graded piece. Cf. [W1] (the notion is also implicit in [Fle]). Our general reference for local cohomology theory is [GrHa].

(7.3) Discussion. Let R be as in (7.2). Let $X = \operatorname{Proj} R$. Then for all $i \geq 2$ we have that $[H^i_m(R)]_t \cong H^{i-1}(X, \mathcal{O}_X(t))$. It follows that if $\dim R \geq 2$ we have that $\mathbf{a}(R) < 0$ if and only if $H^{d-1}(X, \mathcal{O}_X(t))$ vanishes for all $t \geq 0$.

(7.4) Discussion. Now suppose in addition that R is Cohen-Macaulay and that $F_1, \cdots, F_d$ form a homogeneous system of parameters for R. Let $\delta = \sum_{i=1}^d \deg F_i$. Let I be the ideal generated by the F_i.

We note the following facts:

(a) $\mathbf{a}(R) < 0$ if and only if every form of R of degree at least δ is in the ideal generated by the F_i. If R is generated by its one-forms over K, this is equivalent to the condition that $m^{\delta} \subseteq I$. If G is a form of largest degree not in I, then $\mathbf{a}(R) = \deg G - \delta$.

(b) Suppose that $\mathrm{Hilb}_R(z)$ denotes the Hilbert-Poincaré series $\sum_{n=0}^{\infty}(\dim_K[R]_n)z^n$ and that $h(z)$ denotes a rational function whose expansion is this series; then $\mathbf{a}(R) = \deg h(z)$ (i.e., if $h(z)$ is written as $P(z)/Q(z)$ where P, Q are nonzero polynomials in z, then $\mathbf{a}(R) = \deg P(z) - \deg Q(z)$). Thus, $\mathbf{a}(R)$ is determined by the Hilbert function $H_R(n) = \dim_K[R]_n$ (although not by the Hilbert polynomial of R, which agrees with $H_R(n)$ for large n but carries less information).

(c) If R_A denotes an A-free finitely generated $\mathbb{N}$-graded A-algebra over a base ring A with $R_0 = A$, and $A \to K$, $A \to L$ are two homomorphisms from A to fields K, L such that both the fibers $R_K = K \otimes_A R_A$ and $R_L = L \otimes_A R_A$ are Cohen-Macaulay, then $\mathbf{a}(R_K) = \mathbf{a}(R_L)$.

(d) If R has characteristic p then $I = I^F$ if and only if the Frobenius endomorphism acts injectively on $H_m^d(R)$ (recall that I is an ideal generated by a homogeneous system of parameters here). Thus, if $I = I^F$ for one ideal I generated by a homogeneous system of parameters, then the same holds for every ideal generated by a homogeneous system of parameters.

Moreover, in case $I = I^F$, then $\mathbf{a}(R) \le 0$. In this case, if $d = \dim R \ge 2$, then $\mathbf{a}(R) < 0$ if and only if $H^{d-1}(X, \mathscr{O}_X) = 0$, with $(X, \mathscr{O}_X) = \mathrm{Proj}\, R$ as in (7.3).

To understand why (a) holds first note that we may view $H_m^d(R) = H_I^d(R)$ as the cokernel of the map $\bigoplus_{i=1}^d R_{G_i} \to R_F$, where $F = F_1 \cdots F_d$ and G_i denotes $\prod_{j \ne i} F_j$. Thus, $H_m^d(R) = R_F / \sum_{i=1}^d R_{G_i}$. The grading is the obvious one. We may also identify this highest local cohomology module with $\varinjlim_t R/I_t$ where $I_t = (F_1^t, \cdots, F_d^t)$, and the map from R/I_t to R/I_{t+1} is induced by multiplication by F. Because the F's form a regular sequence the maps $R/I_t \to R/I_{t+1}$ are all injective. Under this identification the element represented by r/F^t in $H_m^d(R)$ is sent to the class of r in R/I_t, and is zero if and only if $r \in I_t$. When r is homogeneous the degree of the element represented by the class of r modulo I_t is $\deg r - t\delta$.

Consider a homogeneous element u of $H_m^d(R)$ of largest degree, i.e., of degree equal to $\mathbf{a}(R)$. Since multiplying by any positive degree form of R will kill u, u is in the socle of $H_m^d(R)$. Because R is Cohen-Macaulay, all the modules R/I_t have isomorphic socles, with the isomorphism induced by multiplication by F. Thus, u is represented by a form of highest degree in $R/I = R/I_1$ itself, and the degree of the element represented by G is $\deg G - \delta$. This establishes the final statement in part (a). The first statement is immediate from the final statement, while the second statement follows from the first.

To establish (b), let $d(i) = \deg F_i$, and let $P(z)$ denote the Hilbert-Poincaré series of the ring $R/(F_1, \cdots, F_d)$. $P(z)$ is actually a polynomial and $\deg P(z)$ is the highest degree of a nonzero form of R/I. The Hilbert-Poincaré series of R is then $P(z)/Q(z)$ where $Q(z) = \prod_{i=1}^{d}(1 - z^{d(i)})$, and so $\deg Q(z) = \delta$. The assertion is then immediate from part (a).

(c) is immediate from (b), since R_K and R_L will have the same Hilbert function: $\dim_K [R_K]_n = \dim_L [R_L]_n$, since both are the same as the rank over A of the free A-module $[R_A]_n$.

To establish (d) we first want to observe that if $I = I^F$ then the Frobenius endomorphism acts injectively on $H^d_m(R) = H^d_m(R_m)$. This follows from the theory of F-injective rings developed in [Fe], but we give a brief explanation. We may view $H^d_m(R)$ as the directed union of the modules R/I_t, as in the discussion of (a). The action of the eth iteration of Frobenius sends the element $r + I_t$ to the element $r^q + I_{qt}$. If some element $r + I_t$ is killed then a nonzero element in the socle modulo I_t must be killed. This element will have the form $F^{t-1}s + I_t$ where s represents a nonzero element of the socle modulo I and $F = \prod_i F_i$. But then $F^{q(t-1)}s^q \in I_{qt}$ and it follows that $s^q \in I_{qt} :_R F^{q(t-1)} = I_q = I^{[q]}$, and so $s \in I^F$, a contradiction. On the other hand, if $r^q \in I^{[q]}$ with $q = p^e$ and $r \in R - I$ then the nonzero element $r + I$ in $R/I \subseteq H^d_m(R)$ is killed by the eth iteration of the action of F.

The action of F^e on $H^d_m(R)$ multiplies degrees by p^e, and so an element of positive degree will eventually be mapped into a graded piece of the local cohomology that is zero (the local cohomology has DCC, and so only finitely many positive graded pieces are not zero). Since the action of F is injective, we see at once that $\mathbf{a}(R) \le 0$. Thus, $\mathbf{a}(R) < 0$ if and only if $[H^d_m(R)]_0 = 0$, i.e., by (7.3), if and only if $H^{d-1}(X, \mathcal{O}_X) = 0$, where $X = \mathrm{Proj}(R)$ and $d - 1 = \dim X$. $\square$

(7.5) Discussion. We write $(I)^-$ for the integral closure of the ideal I.

For any ideal I in a Noetherian ring R of characteristic p, $I \subseteq I^F \subseteq I^*$. Moreover, when $\operatorname{ht} I \ge 1$ and I is generated by at most d elements, the generalized Briançon-Skoda theorem of [HH4, §5] shows that $(I^d)^- \subseteq I^*$ and so $I^F + (I^d)^- \subseteq I^*$.

We next record two observations of K. E. Smith.

(7.6) Discussion. Let R be a finitely generated $\mathbf{N}$-graded algebra of Krull dimension $d \geq 1$ over a field K of characteristic p and let $F_1, \cdots, F_d$ be a homogeneous system of parameters for R. Let $\delta = \sum_{i=1}^{d} \deg F_i$. Let $I = (F_1, \cdots, F_d)R$.

(a) (K. E. Smith) If the F_i all have the same degree, then every form of R of degree δ is in $(I^d)^-$, and hence is in I^*.

(b) (K. E. Smith) If R is Cohen-Macaulay, then every homogeneous element of R of degree strictly larger than δ is in I^F, and hence is in I^*.

To see why (a) is true, note that R is module-finite over $K[F_1, \cdots, F_d]$, and choose an equation of integral dependence for a form G of degree δ over $K[F_1, \cdots, F_d]$, say $G^s + P_{s-1}G^{s-1} + \cdots + P_0 = 0$, where $P_j \in K[F_1, \cdots, F_d]$. Let $\deg F_i = \mu$ for all i, so that $\delta = d\mu$. Then the equation still holds when P_{s-j} is replaced by its homogeneous piece of degree $s(\deg G) - (s-j)(\deg G) = j(\deg G) = j\delta = j\,d\mu$, and so we may assume that P_j is homogeneous of degree $j\,d\mu$ in the F's, which have degree μ. But this implies that $P_{s-j} \in ((F_1, \cdots, F_d)^d)^j$, and this in turn implies that G satisfies an equation of integral dependence on the ideal I^d, where $I = (F_1, \cdots, F_d)$. But the generalized Briançon-Skoda theorem of §5 of [HH4] implies that $(I^d)^- \subseteq I^*$ when I is an ideal of height one with at most d generators. □

To see why (b) is true, note that if G is an element of R of degree larger than δ then $G/(F_1 \cdots F_d)$ represents an element of $H^d_m(R)$ of positive degree, say $\eta > 0$. Since the positive graded pieces of $H^d_m(R)$ are all eventually zero, for large q the element $G^q/F_1^q \cdots F_d^q$ represents an element of degree $q\eta$ and so must be zero. But the fact that this element is zero says precisely that $G^q \in (F_1^q, \cdots, F_d^q)$. □

The situation in (a) is somewhat simpler if the ring R is reduced and generated by its one-forms. In that case one sees, just as in the proof of (a) above, that if $G_1, \cdots, G_d$ are a homogeneous system of parameters of degree ν then the integral closure of $(G_1, \cdots, G_d)$ contains m^ν and, using the fact that R is reduced, is equal to m^ν. If $F_1, \cdots, F_d$ form a system of parameters of degree μ then any form of degree μd will be in the integral closure of $(F_1^d, \cdots, F_d^d)$, which is the same as the integral closure of $(F_1, \cdots, F_d)^d$, and, hence, in $(F_1, \cdots, F_d)^*$ by the Briançon-Skoda theorem.

We also want to mention the following related fact (implicit in the proof of (4.9) of [Hu1]).

(c) Let I be an arbitrary ideal of a ring of characteristic p such that I has at most d generators. Then $(I^{d+1})^- \subseteq I^F$.

To see why this is true, let $x \in J^-$, where $J = I^{d+1}$. As follows, for example, from equation (#) in (5.1) of [HH4, p. 44], we have that $x^{k+h} \subseteq J^h$ for some constant positive integer k and all $h \in \mathbb{N}$. Taking $h = ab$ with $a, b \in \mathbb{N}$ we have that $x^{k+ab} \in J^{ab}$ for all $a, b \in \mathbb{N}$. If $a \geq k$ this yields $x^{a+ab} \in J^{ab} = I^{(d+1)ab}$ or $x^{a(b+1)} \in I^{a(d+1)b}$. If $b \geq d$ then $(d+1)b = db + b \geq db + b = (b+1)d$ and $x^{a(b+1)} \in I^{a(b+1)d}$. If $I = (u_1, \cdots, u_d)$ and $I_t = (u_1^t, \cdots, u_d^t)$ then it is easy to see that $I^{td} \subseteq I_t$. Applying this with $t = a(b+1)$ we have that $x^t \in I_t$ whenever $t = a(b+1)$ with $a \geq k$ and $b \geq d$. The desired result now follows if we choose $a = q$, $b = q' - 1$, where q, q' are any two sufficiently large powers of p. $\square$

We are now ready to establish a very useful tool.

(7.7) Theorem. *Let R be a finitely generated $\mathbb{N}$-graded algebra of Krull dimension d over a field K of characteristic p such that $R_0 = K$. Let $F_1, \cdots, F_d$ be forms of positive degree in R constituting a homogeneous system of parameters, and suppose that R is Cohen-Macaulay (equivalently, that the F_i form a regular sequence in R). Suppose also that each of the elements $F_1, \cdots, F_d$ is a test element. Let $I = (F_1, \cdots, F_d)R$. Let δ be the sum of the degrees of the F_j.*

(a) $I^* = I^F + R_\delta$,

where R_δ is the K-vector space of forms of R of degree δ.

(b) *If the F_i all have equal degree then $I^* = I^F + (I^d)^-$.*

(c) *Moreover, I is tightly closed in R if and only if both of the following two conditions hold:*

(1) $I = I^F$.

(2) $\mathbf{a}(R) < 0$.

Proof. (a) By Theorem (4.2b), I^* is homogeneous. It is obvious that $I^F \subseteq I^*$. If the F_i all have equal degree then it follows from (7.6a) that $R_\delta \subseteq I^*$. To see this more generally, choose nonnegative integers $n(1), \cdots, n(d)$ such that the elements $F_i^{n(i)+1}$ all have equal degree, and let δ' be the sum of the degrees of these elements. Let G be an element of R_δ. Then $(F_1^{n(1)} \cdots F_d^{n(d)})G$ has degree δ', and it follows that this element is in $(F_1^{n(1)+1}, \cdots, F_d^{n(d)+1})^*$, so that for fixed $c \in R^\circ$ we have that $cF_1^{n(1)q} \cdots F_d^{n(d)q} G^q \in (F_1^{qn(1)+q}, \cdots, F_d^{qn(d)+q})$ for all $q \gg 0$. Since the F_i are a regular sequence in R, we find that $cG^q \in (F_1^q, \cdots, F_d^q)$ for all $q \gg 0$, so that $G \in I^*$, as required.

To complete the argument it will suffice to show that if G is a form in I^* such that $\deg G \neq \delta$ then $G \in I^F$. If $\deg G > \delta$ then G is in I^F by (7.6b). Thus, we may assume that $\deg G < \delta$.

Since every F_i is a test element, the condition that $G \in I^*$ yields that for every q and every i, $1 \leq i \leq d$, we have $F_i G^q \in I^{[q]}$, and so $IG^q \in I^{[q]}$ and $G^q \in I^{[q]} :_R I$. Since the generators of I form a regular sequence in R, we have that $I^{[q]} :_R I = I^{[q]} + F^{q-1}R$, where $F = F_1 \cdots F_d$. Thus, (#) $G^q = \sum_{j=1}^{d} H_j F_j^q + HF^{q-1}$, and by equating homogeneous components we may assume that all of the terms occurring in the sum are homogeneous of the same degree. We claim that for sufficiently large q we must have that $H = 0$. For if $H \neq 0$ then $\deg(HF^{q-1}) = \deg G^q$ and so, if $H \neq 0$, $\deg H = q(\deg G) - (q-1)(\deg F) = q(\deg G) - (q-1)\delta$, which is negative as soon as q is so large that $(q-1)/q = 1 - 1/q$ is larger than $(\deg G)/\delta$ (note that $\deg G < \delta$ here). But if the HF^{q-1} term is absent from the equation (#) we obtain that $G^q \in I^{[q]}$, as required.

(b) We know from the generalized Briançon-Skoda theorem that $(I^d)^- \subseteq I^*$, so that $I^F + (I^d)^- \subseteq I^*$, and the other inclusion follows from what we have just established together with the fact that $R_\delta \subseteq (I^d)^-$ when the F's all have equal degree, which was proved in (7.6a).

(c) I is tightly closed, from part (a), if and only if I^F and R_δ are both contained in I. The first condition is equivalent to the condition (1) and implies that $\mathbf{a}(R) \leq 0$, by (7.4d), i.e., that all elements of R of degree strictly larger than δ are in I. By (7.4a), $\mathbf{a}(R)$ will then be negative if and only if $R_\delta \subseteq I$. $\square$

(7.8) Remarks. (a) Conditions (1) and (2) are independent of which homogeneous system of parameters we consider. (This is true for condition (1) by (7.4d).) It may be convenient to use one homogeneous system of parameters to test (1) and a different homogeneous system of parameters to calculate $\mathbf{a}(R)$.

(b) The conclusion of (7.7a) is valid under a somewhat weaker hypothesis than that the F_i be test elements for tight closure. For example, if for every ideal J generated by a homogeneous system of parameters, $F_i J^* \subseteq J$. (There may well be an analogue of test element theory for tight closures of parameter ideals. The advantage of restricting attention to parameter ideals is that one expects an increased supply of test elements.)

(7.9) Discussion. Before proceeding further, we want to recall some results on the existence of test elements from [HH3] and [HH9]. Let R be a Noetherian ring of characteristic p. When $R^p = F(R) \subseteq R$ is a module-finite extension, one can define the notion of *strong* F-*regularity* for R. We will not give details here, but we want to recall:

(7.10) Theorem. *Let R be a Noetherian ring of characteristic p such that $R^p \subseteq R$ is module-finite.*

(a) *If R is regular then R is strongly F-regular.*

(b) *If R is strongly F-regular then R and all of its localizations are weakly F-regular, i.e., R is F-regular. Moreover, R is normal and Cohen-Macaulay.*

(c) *If R is Gorenstein then R is weakly F-regular if and only if R is strongly F-regular.*

(d) *If R is reduced and $c \in R^\circ$ is an element such that R_c is strongly F-regular, then c has a power that is a completely stable test element for R. In particular, if R is reduced and $c \in R^\circ$ is an element such that R_c is regular, then c has a power that is a completely stable test element for R.*

Proof. We refer the reader to [HH3, pp. 127–132], especially Theorems (3.1), (3.3), and (3.4), and to §5 of [HH9], especially Theorems (5.5), (5.9), and (5.10), which contain these results. □

The following is part of Theorem (7.32) of [HH9]:

(7.11) Theorem. *Let R be a reduced algebra of finite type over an excellent local ring of characteristic p. If $c \in R^\circ$ is such that R_c is F-regular and Gorenstein (this is equivalent to weakly F-regular and Gorenstein) then c has a power which is a completely stable test element for R.* □

We are now ready to prove

(7.12) Theorem. *Let R be a finitely generated $\mathbb{N}$-graded Cohen-Macaulay K-algebra with homogeneous maximal ideal m such that $R_0 = K$. Suppose that every element of $m \cap R^\circ$ has a power that is a test element. Let $d = \dim R$ and let $X = \mathrm{Proj}(R)$.*

Consider the following conditions:

(1) *There is an ideal generated by a homogeneous system of parameters that is tightly closed.*

(2) *Every ideal generated by part of a homogeneous system of parameters is tightly closed.*

(3) *R_m is F-rational.*

(4) $\mathbf{a}(R) < 0$ *and there is an ideal I generated by a homogeneous system of parameters such that $I = I^F$.*

(5) $H^{d-1}(X, \mathcal{O}_X) = 0$ *and there is an ideal I generated by a homogeneous system of parameters such that $I = I^F$.*

(6) *R is F-rational.*

(7) *R is weakly F-regular.*

Then conditions (1)–(4) *and* (6) *are equivalent, and* (5) *is equivalent as well if* $\dim R \geq 2$. *What is more,* (7) $\Rightarrow$ (6) *and, hence,* (1)–(4) (*and* (5), *if* $\dim R \geq 2$).

If R *is Gorenstein then* (1)–(4), (6), *and* (7) *are all equivalent, and* (5) *as well if* $\dim R \geq 2$.

Proof. The equivalence of (1), (2), (3), and (6) was shown in §4 (as the equivalence of (2), (3), (4), and (1) in Theorem (4.7), as strengthened in the remark preceding Theorem (4.7)). Consider a homogeneous system of parameters consisting of test elements (which exists, since every nonzero element of $m \cap R^{\circ}$ has a power that is a test element). By (7.7c), the ideal I generated by this system of parameters is tightly closed if and only if $\mathbf{a}(R) < 0$ and $I = I^F$. Since the condition $J = J^F$ holds for one ideal generated by a system of parameters if and only if it holds for every such ideal, it follows that (1) and (4) are also equivalent. The equivalence of (4) and (5) when $\dim R \geq 2$ follows from (7.4d). The fact that (7) $\Rightarrow$ (6) in general is obvious.

The statement for the case where R is Gorenstein follows because we know from the equivalence of (1) and (6) in the Gorenstein case in Theorem (4.6) that R is weakly F-regular if and only if R_m is weakly F-regular, and, by Corollary (4.7a) of [HH9], a Gorenstein local ring is weakly F-regular if and only if it is F-rational. □

The following is immediate:

(7.13) Corollary. *Let* R *be a finitely generated* $\mathbb{N}$*-graded Gorenstein* K*-algebra with homogeneous maximal ideal* m *such that* $R_0 = K$. *Let* $d = \dim R$ *and let* $X = \mathrm{Proj}(R)$. *Assume that* $d \geq 2$.

Then the following are equivalent:

(a) R *is F-regular.*

(b) *The localization of* R *at any prime ideal except* m *is F-regular, there exists an ideal* I *generated by a homogeneous system of parameters such that* $I = I^F$, *and* $H^{d-1}(X, \mathcal{O}_X) = 0$ (*equivalently,* $\mathbf{a}(R) < 0$).

Proof. The first hypothesis in (b) coupled with Theorem (7.11) yields that every element of $m \cap R^{\circ}$ has a power that is a test element. We may now apply the equivalence of (7) and (5) or (4) in the Gorenstein case established in Theorem (7.12). □

As an illustration of the usefulness of this result we shall prove:

(7.14) Theorem. *Let* K *be a field of characteristic* p *and let* R *denote either* (1) *the homogeneous coordinate ring of a Grassmann variety or* (2) *the ring obtained by adjoining the entries of a matrix of indeterminates to a field* K *and then killing the ideal of size* t *minors for some fixed* t. *Then* R *is F-regular.*

Proof. In case (2) enlarge the matrix to a square matrix of indeterminates by using some additional rows (or columns, whichever are needed) of indeterminates. Call the ring obtained by killing the size t minors of the large matrix S. We have an obvious map $R \to S$, and killing the additional indeterminates in S yields an obvious algebra retraction $S \to R$. Thus, R is a direct summand, as an R-module, of S. Thus, it suffices to prove the result for S. Hence, we may assume that the matrix is square, and we shall assume this from this point on.

Thus, in both cases, the ring is then known to be Gorenstein. (The homogeneous coordinate ring of a Grassmann variety is known to be Cohen-Macaulay in all characteristics: cf. [Ho8, Lak, Mu], as well as the later treatments in [DEP] and [BrV]. It is also known to be a UFD in all characteristics: see [Sam], and therefore Gorenstein [Mur]. See also Corollary 1.9 on p. 124 of [HR1] and the surrounding discussion in §1.(e). Generic determinantal varieties are known to be Cohen-Macaulay: see [HoE] or [BrV], for example. They are also known to be Gorenstein when the matrix is square: see, for example, Corollary (8.9) on p. 98 of [BrV].) Since we are in the Gorenstein case, F-regularity will follow from weak F-regularity.

In either case it suffices to prove the result when K is algebraically closed, since, if $\overline{K}$ denotes an algebraic closure of K and $\overline{K} \otimes_K R$ is weakly F-regular then R is weakly F-regular. (If S is faithfully flat over R and weakly F-regular so is R: for $I \subseteq R$, $I^* \subseteq (IS)^* \cap R = IS \cap R = I$.)

In the case of the Grassmannian, the localization of the ring at any prime ideal other than the homogeneous maximal ideal m is nonsingular. In the case of the minors, if we localize at any entry of the matrix the resulting ring can be rewritten as a localization of a polynomial ring over the ring obtained by killing $t-1$ size minors of a square matrix of smaller size. Thus, in the second case we may assume by induction that every localization at any prime ideal except the homogeneous maximal ideal m is weakly F-regular.

To complete the proof it will suffice to see in both cases that these rings satisfy $\mathbf{a}(R) < 0$. There are many ways to deduce this, since $\mathbf{a}(R)$ depends only on the Hilbert function, and explicit calculations can be made. Instead, we argue as follows: since the Hilbert function does not depend on the characteristic, for the purpose of computing $\mathbf{a}(R)$ we may assume that the base field has characteristic zero (cf. (7.4c): one may take the base ring A to be $\mathbb{Z}$). In this case, both rings may be obtained as rings of invariants of reductive algebraic groups acting on polynomial rings (cf. [HoE, HR1, We]), and so have rational singularities by virtue of the main result of [Bou]. But then, since there rings are $\mathbb{N}$-graded, the main result of [W1] or [Fle] implies that $\mathbf{a}(R)$ is negative.

It remains to see that in each case there is a homogeneous system of parameters generating an ideal that is contracted with respect to Frobenius. Here we follow a line taken in [FeW]: the key point is that if an associated graded ring of R with respect to a homogeneous ideal is Cohen-Macaulay and has the required property, then so does R (roughly speaking, the property of F-injectivity deforms: cf. the discussion of F-injectivity in [Fe1], or of F-contractedness in [Fe2]). Now each of the rings we are studying is an ordinal Hodge algebra (or algebra with straightening law) with the following property: the associated discrete algebra T is a Cohen-Macaulay ring obtained from a polynomial ring by killing square-free monomials. Such a ring T is automatically F-pure, and by virtue of the sequences of deformations one has that, in both cases, R also has the required property. □

We next want to discuss two examples, both due to Watanabe [W2] and [W3], of F-rational surfaces that are not weakly F-regular. The first is not F-pure. The second is F-pure but is still not weakly F-regular.

(7.15) Example: an F-rational surface that is not F-pure. In [W2] Watanabe discusses surfaces that have rational singularities but are not F-pure. We want to examine a simple example, a quite special case of a family of examples in [W2], and establish its properties using only the techniques of this section. Throughout this example K denotes an algebraically closed field of positive prime characteristic p.

One of the results of [W2] implies that

$$R = K[t, xt^a, x^{-1}t^b, (x+1)^{-1}t^c] \subseteq K(x, t)$$

is not F-pure if a, b, c are positive integers such that $1/a+1/b+1/c < 1$. We want to discuss this ring when $a = b = c = 4$. We want to see, using the results of this section, that $R = K[t, xt^4, x^{-1}t^4, (x+1)^{-1}t^4]$ is F-rational but not F-pure.

Map the polynomial ring $T = K[t, u, v, w]$ onto R as a K-algebra by sending the indeterminates t, u, v, w to $t, xt^4, x^{-1}t^4$, and $(x+1)^{-1}t^4$, respectively. Let P denote the prime ideal which is the kernel of this map. It is easy to see that P contains the size 2 minors of the matrix

$$\begin{bmatrix} t^4 & u & w \\ v & t^4 & v-w \end{bmatrix},$$

i.e., $t^8 - uv$, $t^4(v-w) - vw$, and $u(v-w) - wt^4$. Let J denote the ideal generated by these three minors. Since any two of the minors are relatively prime, this ideal has depth two and so J is a perfect determinantal ideal: it has height (and depth) two and its minimal free resolution is the same as though the entries of the matrix were indeterminates. It follows that R/J is a Cohen-Macaulay ring of dimension two. It is not difficult to verify that $J = P$. The details are left to the reader.

Notice that we can grade R by assigning the variables t, u, v, w degrees $1, 4, 4, 4$ respectively.

If we invert u, v, or $v-w$ we can eliminate one variable (v using the first equation, u using the first equation, or u using the third equation, respectively), and R localized is a localized hypersurface that is easily checked to be nonsingular using the Jacobian criterion in each of the three cases. Since $(u, v, v-w)R = (u, v, w)R$ contains t^8, it is primary to the homogeneous maximal ideal of R. It follows that the Cohen-Macaulay ring R has an isolated singularity at the origin and, so, is normal.

It is trivial to check that if R is a ring of characteristic p and $I_0 \subseteq I$ are ideals of R such that I/I_0 is contracted with respect to Frobenius in R/I_0, then I is contracted with respect to Frobenius in R. Since $R/tR \cong K[u, v, w]/(uv, uw, vw)$ is a ring in which every ideal is contracted with respect to Frobenius (since the ideal that we are killing is generated by square-free monomials in the indeterminates: cf. [HR2, Proposition 5.38, p. 171]), every ideal of R that contains t is contracted with respect to Frobenius. Thus, if I is the ideal of R generated by the parameters $t, u+v+w$, $I = I^F$. Since

$$R/I \cong K[u, v, w]/(uv, uw, vw, u+v+w) \cong K[u, v]/(u^2, uv, v^2),$$

and there are no nonzero elements in degree $\geq 5 = 4+1 = \deg(u+v+w) + \deg t$, we see that $\mathbf{a}(R) < 0$. Thus, R is F-rational by Proposition (7.12).

To finish our discussion of this example, we shall show that in all cases t^7 is in the Frobenius closure of $(u, v)R$, although it is clear from our presentation of R that $t^7 \notin (u, v)R$. What we want to show is that when the characteristic is p, then $t^{7p} \in (u^p, v^p)R$. In fact, we shall show something much stronger: in every characteristic (or even if we worked over $\mathbb{Z}$ as a base ring), $t^{7n} \in (u^n, v^n)R$ for every positive integer $n \geq 2$. (Calculations with the program Macaulay helped to convince us that this was true.)

To prove this, we must show that t^{7n} is a K-linear combination of monomials $t^\alpha u^{\beta+n} v^\gamma w^\delta$ or $t^\alpha u^\beta v^{\gamma+n} w^\delta$ where, in each term occurring, $\alpha + 4(\beta+\gamma+\delta+n) = 7n$ and $\alpha, \beta, \gamma \in \mathbb{N}$. Thus, $4(\beta+\gamma+\delta) \leq 3n$

and $\alpha = 3n - 4(\beta + \gamma + \delta)$. We recall our original description of R, as $K[t, xt^4, x^{-1}t^4, (x+1)^{-1}t^4]$. After taking out the factor of t^{7n} that occurs in every term, we see that an equivalent problem is to show that 1 is in the K-span of the elements in the set $\{x^n, x^{-n}\}\cdot\{x^\alpha x^{-\beta}(x+1)^{-\gamma} : \alpha+\beta+\gamma \le k\}$ where k is the integer part of $3n/4$, and $A\cdot B$ denotes $\{ab : a \in A \text{ and } b \in B\}$. After multiplying all terms by $(x+1)^k$, we see that it suffices to show that $(x+1)^k$ is in the K-span V of the elements of $\{x^n, x^{-n}\}\cdot\{x^{\alpha-\beta}(x+1)^{k-\gamma} : \alpha+\beta+\gamma \le k\}$.

Taking $\alpha = \beta = 0$, $\gamma = k, k-1, \cdots, 0$ successively, and multiplying each time by x^{-n}, we see that $x^{-n} \in V$, then that $x^{-(n-1)} \in V, \ldots$ and so forth, until we obtain that $x^{-(n-k)} \in V$. Then taking $\beta = \gamma = 0$, $\alpha = 1, \cdots, k$, and multiplying each time by x^{-n}, we find successively that $x^{-(n-k-1)}, \cdots, x^{2k-n}$ are in V. In particular, $1, x, \cdots, x^{2k-n} \in V$.

On the other hand, taking $\alpha = \beta = 0$, $\gamma = k, k-1, \cdots, 0$ successively, and multiplying each time by x^n, we find that $x^n, \cdots, x^{n+k-1} \in V$. Taking $\alpha = \gamma = 0$, $\beta = 1, 2, \cdots, k$ successively, and multiplying each time by x^n, we find that $x^{n-1}, \cdots, x^{n-k} \in V$. Thus, so long as $2k-n \ge n-k-1$, i.e., so long as (#) $3k \ge 2n-1$, we see that all of the powers of x from 1 to x^{n+k-1} are in V, and so $(x+1)^k$ is in V. Since $k \ge 3n/4 - 3/4$, we see at once that the required inequality (#) holds provided that $n \ge 5$. However, if $n = 2, 3$, or 4 (so that $k = 1, 2$, or 3, respectively) the required inequality (#) also holds. This completes the proof that $(x+1)^k \in V$ in all cases when $n \ge 2$, so that $(t^7)^n \in (u^n, v^n)R$ for all $n \ge 2$, and also completes the proof that R is not F-pure. □

(7.16) An F-rational, F-pure surface that is not weakly F-regular. In [W3] Watanabe gives an example of an F-rational, F-pure ring that is not F-regular. We want to explore the nature of this example without using the machinery of [W3]. Let $S = K[X, Y, Z]/(F)$ where K is an algebraically closed (for simplicity) field of characteristic p with $p \equiv 1$ modulo 3, and $F = X^3 - YZ(Y+Z)$. Let ω be a primitive cube root of unity in K. Let $G = \{1, \omega, \omega^2\}$ act K-linearly on S so as to send the images x, y, z of X, Y, Z to $x, \omega y, \omega z$. Let $R = S^G$ be the fixed ring of this action, which is generated over K by x, y^3, y^2z, and z^3 (note that $yz^2 = x^3 - y^2z$).

We first observe that S is F-pure. Because of the grading, it suffices to check this at the origin, and because the ring is Gorenstein it suffices to check that $(y, z)S$ is contracted with respect to Frobenius, i.e. that $(x^2)^p$ is not in $(y^p, z^p)S$. But if $p = 3k+1$ then $x^{2p} = (x^3)^{2k}x^2 = y^{2k}z^{2k}(y+z)^{2k}x^2$, and since $1, x, x^2$ is a free basis for S

over $B = K[y, z]$ it follows that this element is in $(y^p, z^p)S$ if and only if $y^{2k}z^{2k}(y+z)^{2k}$ is in $(y^p, z^p)B = (y^{3k+1}, z^{3k+1})B$. The key point is that the "center" term $\binom{2k}{k}y^k z^k$ in the expansion of $(y+z)^{2k}$ yields a nonzero monomial term $\binom{2k}{k}y^{3k}z^{3k}$ that is not in $(x^{3k+1}, y^{3k+1})B$.

On the other hand, by repeating this calculation with $q = p^e$ replacing p, we see that $y(x^2)^q$ (or $z(x^2)^q$) is in $(y^q, z^q)S$ for all $q = p^e \geq p$, and so $x^2 \in ((y, z)S)^*$.

Because $R \to S$ splits over R and S is F-pure, R is F-pure.

It is easy to check that S has an isolated singularity at the origin, and so is normal, and it follows that R is normal with an isolated singularity at the origin as well. Therefore, every homogeneous element of R of positive degree has a power that is a test element. It follows from Proposition (7.12) that R will be F-rational provided that $\mathbf{a}(R) < 0$. Since R is F-pure, it is clear that $\mathbf{a}(R) \leq 0$. Since y^3, z^3 is a system of parameters for R, it will suffice to see that no nonzero elements of degree $6 = 3 + 3$ survive in $R/(y^3, z^3)$. But in degree $3h$, where h is an integer, every element of R can be expressed as polynomial over k in degree 3 elements of $K[y, z]$ and powers of x that must have degree divisible by 3. The defining relation of S can be used to rewrite these powers of x in terms of y, z. Thus, the degree 6 part of R is spanned by the monomials $y^i z^j$ where $i + j = 6$, and these are all in $(y^3, z^3)R$.

On the other hand, since x^2 is in the tight closure of $(y, z)S$, it follows that $y^3z^3x^2$ is in the tight closure of $(y^4z^3, y^3z^4)S$ in S, and hence in the tight closure of the larger ideal $(y^4z^2, y^2z^4)S$. But then $y^3z^3x^2$ is in the tight closure of $(y^4z^2, y^2z^4)R$ in R: since S is module-finite over R we may apply Corollary 5.23. But $y^3z^3x^2$ is not in $(y^4z^2, y^2z^4)R$ (nor in its expansion to S: we have that $1, x, x^2$ is a free basis for S over $B = K[y, z]$ and $y^3z^3 \notin (y^4z^2, y^2z^4)B$). Thus, R is not weakly F-regular, although it is F-rational and F-pure. $\square$

8. Integral closures of ideals and test elements

In this section we discuss again the theory of integral closures of ideals from the point of view of §5 of [HH4], where a new proof (in characteristic p) of the Briançon-Skoda theorem, in a generalized form, was given. (The equal characteristic 0 case will be treated in [HH10].) In particular, we obtain a new generalization of the Briançon-Skoda theorem for Cohen-Macaulay rings with isolated singularities. (See Theorem (8.10) and the discussion in (8.12).) Our results depend heavily on the theory of test elements developed in §6 of [HH4] and pushed much further in §6 of [HH9]. In consequence, we shall need to impose some restrictions on the rings to guarantee the existence of sufficiently many test elements. A number of results depend on the size of the test ideal $\tau(R)$ (or a modification of it, $\tau_{\mathrm{par}}(R)$, discussed in (8.7)). We conclude this section with a theorem, based on a result of Lipman and Sathaye [LS] and the theory of §6 of [HH4], that allows one to calculate easily a large number of elements of R that are in $\tau(R)$.

We also prove a theorem of Itoh and Huneke using tight closure techniques (cf. [It1, It2, Hu1]). To wit, we show that if the local rings of a Noetherian ring R have equidimensional completions and I is a parameter ideal of R then $I^n \cap (I^{n+1})^- = (I^n)(I^-)$ for all $n \geq 1$, where J^- denotes the integral closure of the ideal J. Moreover, we obtain an analogue of the above statement with integral closure replaced by tight closure.

In [Hu3] a "uniform" Briançon-Skoda theorem (Theorem 4.13) is obtained for a large class of rings (not necessarily of characteristic p). Several of the results we obtain here are closely related: they show that in proving results concerning when $(I^{k+d+T})^- \subseteq I^k$ for all k (where d is the dimension of the ring and T is independent of I) the size of T can be bounded from data about the test ideal.

Other related results may be found in [Sw1] and [Sw2].

We begin by recalling a version of the Briançon-Skoda theorem (Theorem (5.4)) proved in §5 of [HH4], with some improvement:

(8.1) Theorem (generalized Briançon-Skoda theorem). *Let R be a Noetherian ring of characteristic p, and let I be an ideal of R generated by d elements. Then for all $k \in \mathbb{N}$, $(I^{d+k})^- \subseteq (I^k)^*$.*

Proof. If I has positive height this is precisely the result of Theorem (5.4) of [HH1]. In the general case, suppose that $x \in (I^{d+k})^-$. To show that $x \in (I^k)^*$, it suffices to verify this modulo every minimal prime $\mathfrak{p}$ of R, by Proposition (6.25a) of [HH4]. When we pass to $R/\mathfrak{p}$, the hypothesis that x is in the integral closure of $I^{d+k}(R/\mathfrak{p})$ is preserved. Thus, we are in the domain case, where I is either (0) (in which case the result is trivial) or has positive height. □

(8.2) Discussion. Let I be an ideal of a Noetherian ring R (not necessarily of characteristic p). Let $S = \mathrm{gr}_I R$ so that $S_j = I^j/I^{j+1}$. We observe that the following two conditions are equivalent:

(1) $I^{s+t} : I^t = I^s$ for all $s, t \in \mathbb{N}$.

(2) The ideal $S_+ = \bigoplus_{j=1}^{\infty} S_j$ has positive depth (i.e., contains a nonzerodivisor) in S.

To see this, first assume (2). Evidently, $I^{s+t} : I^t \supseteq I^s$. If $u \in I^{s+t} : I^t - I^s$ choose $h \in \mathbb{N}$ so that $u \in I^h - I^{h+1}$, and note that $h < s$. Then if $\overline{u} \neq 0$ is the image of u in S_h we see that $\overline{u}(S_t) = 0$ in S, and S_t generates $(S_+)^t$ in S, so that $\overline{u}$ kills a power of S_+ in S. It follows that $\operatorname{depth} S_+ = 0$, a contradiction.

On the other hand, if $\operatorname{depth} S_+ = 0$ we can choose a nonzero form $\overline{u}$ of degree h which kills it, where $u \in I^h - I^{h+1}$. But then $\overline{u}(S_1) = 0$ implies that $uI \subseteq I^{h+2}$, and so $u \in I^{h+2} : I$ which is contained in I^{h+1} if (1) holds. Thus, (1) $\Rightarrow$ (2) holds as well.

The equivalent conditions (1) and (2) hold, in particular, whenever I is generated locally by a regular sequence or when $gr_I R$ is a domain and $I \neq I^2$.

(8.3) Discussion. Let R be a Noetherian ring of positive characteristic p. Recall that $\tau(R)$ denotes the ideal $\bigcap_M \operatorname{Ann}_R(0^*_M)$, where M runs through (representatives of the isomorphism classes of) all finitely generated R-modules. The ring R has a test element if and only if $\tau(R)$ meets R°, in which case $\tau(R)$ is the ideal generated by the test elements of R. See Definition (8.22) and Proposition (8.23) of [HH4].

We can now prove:

(8.4) Theorem. *Let R be a Noetherian ring of characteristic p and let I be an ideal of R generated by at most d elements such that the equivalent conditions* (1) *and* (2) *of* (8.2) *hold.*

Suppose that t is a nonnegative integer such that $I^t \subseteq \tau(R)$. Then for all $k \geq 0$, $(I^{d+t+k})^- \subseteq I^k$.

Proof. $(I^{d+t+k})^- \subseteq (I^{k+t})^*$ by Theorem (8.1), and since $I^t \subseteq \tau(R)$ we have that $I^t(I^{d+t+k})^- \subseteq I^{k+t}$ and $(I^{d+t+k})^- \subseteq (I^{k+t}) :_R I^t = I^k$. □

(8.5) Remark. The conclusion of (8.4) is evidently valid (with the same proof) if one replaces $\tau(R)$ by a (possibly larger) ideal J of R with the property that $J((I^n)^*) \subseteq I^n$ for all $n \in \mathbb{N}$.

In the sequel we shall study the situation where one has a local ring with an m-primary ideal of test elements. We first want to note several situations where this occurs.

(8.6) Proposition. *Let (R, m, K) be a reduced Noetherian local ring. Then $\tau(R)$ is m-primary in each of the following situations:*

(a) *R is excellent, and for all primes $P \neq m$, R_P is F-regular and Gorenstein.*

(b) *R is excellent, with an isolated singularity.*

(c) *$F: R \to R$ is module-finite, and for all primes $P \neq m$, R_P is strongly F-regular.*

Proof. Part (b) follows from part (a), while (a) and (c) are immediate from [HH9, Corollary (7.34); HH9, Theorem (5.10)], respectively. $\square$

(8.7) Definition. Let R be an equidimensional local ring of characteristic p. We define $\tau_{\mathrm{par}}(R)$ to be $\bigcap_I (I :_R I^*)$, where I runs through all ideals generated by a system of parameters. Roughly speaking, the elements in $\tau_{\mathrm{par}}(R)$ that are in R° are test elements for parameter tight closure. We are mainly interested here in the case where R is Cohen-Macaulay.

(8.8) Theorem. *Let (R, m, K) be an excellent equidimensional local ring of characteristic p.*

(a) *If $c \in R$, then $c \in \tau_{\mathrm{par}}(R)$ if and only if for every ideal I generated by part (or all) of a system of parameters, for all $x \in R$ we have that $x \in I^*$ implies $cx^q \in I^{[q]}$ for all $q = p^e$.*

(b) *If R is Cohen-Macaulay, $x_1, \cdots, x_d$ is a system of parameters, and $I_t = (x_1^t, \cdots, x_d^t)R$, then $\tau_{\mathrm{par}}(R) = \bigcap_t I_t :_R I_t^*$.*

(c) *If R is Cohen-Macaulay and I is any ideal generated by monomials in a system of parameters, then $\tau_{\mathrm{par}}(R)(I^*) \subseteq I$. Moreover, if R is Cohen-Macaulay, $c \in \tau_{\mathrm{par}}(R)$, I is an ideal generated by monomials in a system of parameters and $z \in I^*$, then $cz^q \in I^{[q]}$ for all $q = p^e$.*

Proof. (a) The "if" part is immediate: the condition applies for $q = 0$ when I is generated by a full system of parameters, and then asserts, in particular, that $cI^* \subseteq I$. Now suppose that $cJ^* \subseteq J$ whenever J is generated by a full system of parameters, and that $I = (x_1, \cdots, x_h)R$ is generated by part of a system of parameters. Let $x_1, \cdots, x_d$ be a full system of parameters. It suffices to show that if $z \in I^*$ then $cz^q \in (x_1^q, \cdots, x_h^q)$ for all q. But the hypothesis implies that $z^q \in (x_1^q, \cdots, x_h^q)^*$ and hence to $(x_1^q, \cdots, x_h^q, x_{h+1}^N, \cdots, x_d^N)^*$ for all positive integers N. But then cz^q is in $(x_1^q, \cdots, x_h^q, x_{h+1}^N, \cdots, x_d^N)$ for all positive integers N (this ideal is generated by a full system of parameters) and if we intersect these ideals for fixed q as N varies we obtain that $cz^q \in (x_1^q, \cdots, x_h^q)$.

(b) It is clear that $\tau_{\mathrm{par}}(R)$ is contained in $\bigcap_t I_t :_R I_t^*$. Now suppose that c is in the intersection of the $I_t :_R I_t^*$ and let J be any ideal generated by a system of parameters. The highest local cohomology module of R with support in the maximal ideal may be represented as an increasing union of submodules of the form R/I_t, and a similar construction may be performed with J replacing I. Thus, for all $t \gg 0$, there is an injection $R/J \to R/I_t$. Then J^*/J (the tight closure of 0 in R/J) is contained in I_t^*/I_t (the tight closure of 0 in R/I_t) for large t, and so

$I_t :_R I_t^* = \mathrm{Ann}_R(I_t^*/I_t) \subseteq \mathrm{Ann}_R(J^*/J) = J :_R J^*$ for $t \gg 0$. It follows that $cJ^* \subseteq J$, as required.

(c) The second assertion follows from the first, because every $I^{[q]}$ is generated by monomials in a system of parameters. Thus, it suffices to prove the first assertion.

Let Q be any ideal generated by monomials in a system of parameters. Since R is C-M local, a system of parameters is a permutable R-sequence, and so the results of [EH] imply that we may write Q as a finite intersection $Q_1 \cap \cdots \cap Q_h$ where every Q_j is generated by powers of the elements in a certain subset of the parameters. Thus, every Q_j is generated by part of a system of parameters. Now suppose that $z \in Q^*$. Then z is in every Q_j^*, and so cz is in every Q_j and, hence, in Q. □

(8.9) Remarks. (a) It is desirable to define $\tau_{\mathrm{par}}(R)$ whether R is local or not and whether R is equidimensional or not. One natural candidate is to take the intersection of all the ideals $I :_R I^*$ where I is an ideal generated by h elements for some integer h and such that $\mathrm{mnht}\, I = h$. The reader is referred to [Sm] and [Ve1] where the ideal of test elements for parameter tight closure and other related ideas are explored in greater detail.

(b) Let J be any ideal of R that is an intersection (possibly infinite) of ideals generated by subsets of a system of parameters. The argument in the proof of (8.7c) shows that if $c \in \tau_{\mathrm{par}}(R)$ then $cJ^* \subseteq J$. Thus, it is natural to ask which ideals of R are intersections of ideals generated by subsets of systems of parameters. The issue is considered in [Gla].

(8.10) Theorem. *Let (R, m, K) be a Cohen-Macaulay local ring of dimension d such that $\tau(R)$ is m-primary (or such that $\tau_{\mathrm{par}}(R)$ is m-primary). Assume either that K is infinite or that R is excellent.*

If $I \subseteq m$ is an ideal and $t \in \mathbb{N}$ is such that $I^t \subseteq \tau(R)$ (or $\tau_{\mathrm{par}}(R)$), then $(I^{d+t+k})^- \subseteq I^k$ for all $k \geq 0$.

In particular, if $m^T \subseteq \tau_{\mathrm{par}}(R)$ then for every ideal I of R, $(I^{d+T+k})^- \subseteq I^k$ for all $k \geq 0$.

Proof. Since $\tau(R) \subseteq \tau_{\mathrm{par}}(R)$, it suffices to work with $\tau_{\mathrm{par}}(R)$. Suppose we have shown the theorem when I is m-primary. Choose h so large that $m^h \subseteq \tau_{\mathrm{par}}(R)$. Then $(I^{d+t+k})^- \subseteq \bigcap_{n=h}^{\infty}((I+m^n)^{d+t+k})^- \subseteq \bigcap_{n=h}^{\infty}(I+m^n)^k \subseteq \bigcap_{n=h}^{\infty}(I^k + m^n) = I^k$, where the second inclusion is obtained by applying the result for m-primary ideals to $I + m^n$. Thus, we may assume that I is m-primary.

We next explain how to complete the argument when K is infinite.

In this case I is integral over an ideal $I_0 \subseteq I$ generated by a system of parameters for R (cf. [NR]), a so-called *minimal reduction* of I. It suffices to prove the result for I_0, since $(I^{d+t+k})^- = (I_0^{d+t+k})^- \subseteq I_0^k \subseteq I^k$ (note that since $I^t \subseteq \tau_{\text{par}}(R)$, we also have that $I_0^t \subseteq \tau_{\text{par}}(R)$).

Thus, we may assume without loss of generality that I is generated by a system of parameters for R. Since R is Cohen-Macaulay, the system of parameters is an R-sequence and we may apply Theorem (8.4).

If K is finite and R is excellent we modify the argument as follows: replace R by $R(z)$, where z is an indeterminate (this is the localization of $R[z]$ at $mR[z]$). We may calculate $\tau_{\text{par}}(R(z))$ as $\bigcap_s J_sR(z) :_{R(z)} (J_sR(z))^*$, where J_s is generated by the sth powers of a fixed system of parameters for R. By Theorem (7.16a) of [HH9] applied with $S = R(z)$, we have that $(J_sR(z))^* = (J_s^*)R(z)$, so that $\tau_{\text{par}}(R(z)) = \bigcap_s(J_s :_R J_s^*)R(z)$. Since R/J_s embeds in R/J_{s+1}, the sequence $J_s :_R J_s^*$ is a decreasing sequence of ideals. Since the intersection $\tau_{\text{par}}(R)$ is m-primary, it follows from the fact that $R/\tau_{\text{par}}(R)$ has DCC that the sequence $J_s :_R J_s^*$ is eventually constant, and the constant value must be $\tau_{\text{par}}(R)$. But this shows that $\tau_{\text{par}}(R(z)) = (\tau_{\text{par}}(R))R(z)$. We may therefore apply the result for $R(z)$ (which has an infinite residue field) to conclude that $((IR(z))^{d+t+k})^- \subseteq (IR(z))^k = I^kR(z)$. It follows that $(I^{d+t+k})^- \subseteq I^kR(z) \cap R = I^k$. □

(8.11) Remark. Instead of assuming that R is Cohen-Macaulay, we may assume that for every ideal I generated by a system of parameters, the ideal generated by I/I^2 in $\mathrm{gr}_I R$ has positive depth. The result is then valid for $\tau(R)$, but $\tau_{\text{par}}(R)$ should be replaced by $\bigcap I : I^*$ for all powers I of ideals generated by systems of parameters.

(8.12) The case of an isolated excellent Cohen-Macaulay singularity. Theorem (8.10) applies to the case of an excellent Cohen-Macaulay ring (R, m, K) of dimension q with an isolated singularity: we then know that $\tau(R)$ is m-primary. Theorem (8.10) shows that there is a positive integer T *independent* of I such that for all ideals I of R, $(I^{d+k+T})^- \subseteq I^k$ for every integer $k \in \mathbb{N}$. Theorem (4.13) of [Hu3] is quite similar and in many ways more general, but does not offer explicit control of T.

We note that D. Rees has shown [Re] that if R is analytically unramified then there is a positive integer T *depending on* I such that $(I^{d+k+T})^- \subseteq I^k$ for all $k \in \mathbb{N}$.

Observe that when $I \subseteq m^T$, Theorem (8.10) shows that $(I^{d+1+k})^- \subseteq I^k$ for all $k \in \mathbb{N}$, so that for ideals "deep" enough inside R we have a statement that is only a slight weakening of the original Briançon-Skoda theorem.

Finally, we also note that Theorem (8.10) gives very explicit bounds if we can compute $\tau(R)$, $\tau_{\text{par}}(R)$, or any sufficiently large ideal inside $\tau_{\text{par}}(R)$. A method of showing that elements of R are in $\tau(R)$ is given in Theorem (8.23).

The next result, while quite easy, is sufficiently useful to be worth recording. Recall that the Frobenius closure N^F of N in M consists of all elements $u \in M$ such that $u^q \in N^{[q]}$ for some $q = p^e$.

(8.13) Proposition. *Let R be a Noetherian ring such that $\tau(R)$ contains a power of an ideal I. Let $N \subseteq M$ be finitely generated R-modules and $N^* = N^*_M$, $N^F = N^F_M$.*

(a) *Then N^*/N^F is killed by I. (Note that if I is maximal so that $R/I = K$, a field, then it is a K-vector space.)*

(b) *If, moreover, R is F-pure, then $\tau(R)$ is radical and so contains I, and N^*/N is killed by I.*

Proof. (a) Let $u \in N^*$ and $c \in I$. Then $c^q \in \tau(R)$ for some choice of q, and so $c^q u^q \in N^{[q]}$ (for $u \in N^* \Rightarrow u^q \in (N^{[q]})^*$ for all q). But then $cu \in N^F$, as required.

(b) Under the additional hypothesis of F-purity we have that $N^F = N$ for every $N \subseteq M$, and so every N^*/N is killed by I. This shows also that $\tau(R) \supseteq I$, and we may choose I to be the radical of $\tau(R)$. □

(8.14) Corollary. *Let (R, m, K) be Cohen-Macaulay with an m-primary test ideal such that K is infinite or R is excellent.*

Suppose either that R is F-pure, that $\tau(R) \supseteq m$, or even that $\tau_{\text{par}}(R) \supseteq m$.

Then $(I^{d+1+k})^- \subseteq I^k$ for all $I \subseteq R$ and all $k \in \mathbb{N}$.

Proof. By (8.13) F-purity guarantees that $\tau(R) \supseteq m$, and we may then apply Theorem (8.10) with $T = 1$. □

We note that a special case of (8.14) was done in (4.9) of [Hu1].

(8.15) Theorem. *Let (R, m, K) be a Cohen-Macaulay local ring with $\dim R = d$, let $J = \tau_{\text{par}}(R)$ be m-primary, let $h = l(R/J)$, where l denotes length, and let θ be the type of R. Let $I = (x_1, \cdots, x_d)R$ be an ideal generated by a system of parameters for R. If $t \in \mathbb{N}$, let $I_t = (x_1^t, \cdots, x_d^t)$. If $\mathbf{t} = (t_1, \cdots, t_d) \in (\mathbb{Z}^+)^d$ (where $\mathbb{Z}^+ = \mathbb{N} - \{0\}$), let $I_{\mathbf{t}} = (x_1^{t_1}, \cdots, x_d^{t_d})$. If $\mathbf{t} \in \mathbb{N}^d$ let $x^{\mathbf{t}}$ denote the product $x_1^{t_1} \cdots x_d^{t_d}$. Note that if $\mathbf{s}, \mathbf{t} \in (\mathbb{Z}^+)^d$ with $t \geq s$ (i.e., $\mathbf{t} - \mathbf{s} \in \mathbb{N}^d$) then multiplication by $x^{\mathbf{t}-\mathbf{s}}$ induces an injection of $R/I_{\mathbf{s}}$ into $R/I_{\mathbf{t}}$ that takes I_s^* into I_t^*.*

(a) $l(I^*/I) \leq \theta h$.

(b) *There exists* $\mathbf{s} \in \mathbb{N}^d$ *such that for all* $\mathbf{u} \geq \mathbf{t} \geq \mathbf{s}$ *multiplication by* $x^{\mathbf{u}-\mathbf{t}}$ *induces an isomorphism of* $I_{\mathbf{t}}^*/I_{\mathbf{t}}$ *with* $I_{\mathbf{u}}^*/I_{\mathbf{u}}$.

Proof. (a) I^*/I is contained in R/I, which has a socle of dimension (as a K-vector space) θ, and is killed by J so that it may be viewed as an R/J-module. It embeds in its injective hull E as an (R/J)-module, which will be the direct sum of at most θ copies of the injective hull E_0 of K over R/J. Thus, $l(I^*/I) \leq l(E_0^\theta) = \theta l(E_0) = \theta l(R/J) = \theta h$.

(b) Multiplication by $x^{\mathbf{u}-\mathbf{t}}$ embeds $I_{\mathbf{t}}^*/I_{\mathbf{t}}$ in $I_{\mathbf{u}}^*/I_{\mathbf{u}}$ when $\mathbf{u} \geq \mathbf{t}$. Hence, $l(I_{\mathbf{t}}^*/I_{\mathbf{t}})$ is a nondecreasing function of $\mathbf{t} \in (\mathbb{Z}^+)^d$ bounded by θh. It follows that it is eventually constant for all $\mathbf{t} \geq \mathbf{s}$ for some $\mathbf{s} \in (\mathbb{Z}^+)^d$, and hence that the injections $I_{\mathbf{t}}^*/I_{\mathbf{t}} \to I_{\mathbf{u}}^*/I_{\mathbf{u}}$ are isomorphisms for $\mathbf{u} \geq \mathbf{t} \geq \mathbf{s}$. $\square$

We next give a tight closure proof of a theorem of Itoh and, independently, Huneke (cf. [It1], [It2], [Hu1]). We note that Itoh's proof gives the result even in mixed characteristic.

(8.16) Theorem. *Let R be a Noetherian ring of characteristic p such that the completion of R at every maximal ideal is equidimensional. Let I be an ideal of R such that for every prime P of R with $P \subseteq I$, IR_P is generated by part of a system of parameters. Then for every $n \in \mathbb{N}$,*

$$I^n \cap (I^{n+1})^- = I^n(I^-).$$

Note that the left-hand side obviously contains the right-hand side. If the other inclusion fails, this is preserved upon localization at a suitable maximal ideal. Thus, there is no loss of generality in assuming that I is generated by elements $x_1, \cdots, x_d \in R$ whose images in R_P form part of a system of parameters for every prime ideal P of R containing I. The following result is then sharper.

(8.17) Theorem. *Let R be a Noetherian ring of characteristic p such that the completion of R at every maximal ideal is equidimensional. Let $x_1, \cdots, x_d$ be elements of R, let $I = (x_1, \cdots, x_d)R$, and suppose that the images of the x_i form part of a system of parameters in R_P for every prime ideal P containing I. Let F be a homogeneous polynomial of degree n in d variables with coefficients in R. Then if $F(x_1, \cdots, x_d) \in (I^{n+1})^-$, every coefficient of F is in I^-.*

Hence, $I^n \cap (I^{n+1})^- = (I^n)I^-$.

Proof. Since every element of I^n can be represented as $F(x_1, \cdots, x_d)$ for some choice of homogeneous polynomial F of degree n, the final statement (hence, also, (8.16)) is immediate from the statement of the first paragraph. If the result fails we can choose a monomial μ of degree n such that its coefficient r_μ in F is not in I^-. We may replace R by its localization at a suitable maximal ideal without affecting the situation, and by Lemma (8.18) below we may replace R by its completion (the image of r_μ will still not be in I^-). Thus, we may assume that the ring is complete local and equidimensional, and that $x_1, \cdots, x_d$ is part of a system of parameters. Let J be the ideal of R generated by all of the monomials in the x's of degree n other than μ. Since $y = F(x_1, \cdots, x_d) \in (I^{n+1})^-$, there exists a positive integer h such that $(I^{n+1} + Ry)^{h+N} = (I^{n+1})^{N+1}(I^{n+1} + Ry)^{h-1}$ for all N. (See, for example, condition (#) in (5.1) of [HH4].) In particular, $y^{h+N} \in (I^{n+1})^{N+1}$ for all h. Choose c, e.g., a monomial of degree at least $(n+1)(h-1)$ in the x's, such that $c \in R^\circ$ and $c \in (I^{n+1})^{(h-1)}$. Then $cy^{h+N} \in (I^{n+1})^{h+N}$ for all N, and it follows in particular that if q is any power of p with $q > h$ then $cy^q \in I^{(n+1)q}$. Now, y is the sum of $r_\mu \mu$ and an element of J, so that $cy^q - cr_\mu^q \mu^q \in J^{[q]}$. It follows that for all $q > h$, $cr_\mu^q \mu^q \in I^{(n+1)q} + J^{[q]}$, and so $cr_\mu^q \in (I^{(n+1)q} + J^{[q]}) :_R \mu^q$ for all $q > h$. By Theorem (7.9) of [HH4] the colon ideal on the right is contained in the tight closure of the "formal colon ideal" (calculated as though the x_i were a regular sequence). In the formal calculation, colon distributes over sum. Moreover, in the formal case $I^{(n+1)q} : \mu^q$ is in I^q (since μ^q has degree nq) and $J^{[q]} : \mu^q$ is in $I^{[q]}$. Thus, $cr_\mu^q \in (I^q)^*$ for all $q > h$, and $(I^q)^* \subseteq (I^q)^-$. If r_μ is not in I^- there is a discrete valuation v of R infinite only on a minimal prime of R such that $v(r_\mu) < \min\{v(u) : u \in I\}$. But then $v(c) + qv(r_\mu) \geq \min\{v(u) : u \in (I^q)^*\} \geq \min\{v(u) : u \in I^q\} = q \cdot \min\{v(u) : u \in I\}$, for all $q > h$. Dividing by q and taking the limit as $q \to \infty$ we obtain a contradiction. □

(8.18) Lemma. *Let $R \subseteq S$ be such that ideals of R are contracted from S (which holds if $R \to S$ is faithfully flat or pure). Let I be an ideal of R. Then $(IS)^- \cap R = I^-$.*

Proof. It is clear that $(IS)^- \cap R \supseteq I^-$. But if $y \in R$ and is in $(IS)^-$ then for some $k \in \mathbb{Z}^+$, $y^k \in IS(IS + yS)^{k-1} = I(I + yR)^{k-1}S$, and the contraction of the latter ideal to R will be $I(I + yR)^{k-1}$, which shows that $y \in I^-$. □

(8.19) Remark. The conclusion of Theorem (8.17) is valid whenever $x_1, \cdots, x_d$ are elements of R such that the colon of two monomial ideals in the x's is contained in the tight closure of the "formal colon ideal" (i.e., the ideal generated by those monomials in the x's which are in the colon when the x's are replaced by indeterminates), or if this can be shown to be true after completion. E.g., we may drop the equidimensionality condition on R if we assume instead that the x_i are part of a system of parameters modulo every minimal prime of the completion of R at a maximal ideal containing I. If R is universally catenary, it suffices if $\mathrm{mnht}(x_1, \cdots, x_d)R \geq d$.

We next prove an analogue of (8.16) for tight closure.

(8.20) Theorem. *Let R be a Noetherian ring of characteristic p, let $I = (x_1, \cdots, x_d)R$, and suppose that at least one of the following two conditions holds:*

(a) *R has a completely stable weak test element, and for every maximal ideal P of R with $P \supseteq I$, the completion of R at P is equidimensional and the images of $x_1, \cdots, x_d$ in R_P are part of a system of parameters.*

(b) *R has a weak test element, and the x's satisfy the condition that the colon of two monomial ideals in the x's is contained in the tight closure of the formal colon ideal.*

Then $I^n \cap (I^{n+1})^* = I^n(I^*)$.

Moreover, if F is a homogeneous polynomial of degree n in d variables such that $F(x) \in (I^{n+1})^*$ then all coefficients of F are in I^*.

Proof. First note that it suffices to prove the final statement. Second, in case (a) if one has a counterexample one can preserve that a certain coefficient of F is outside I^* while localizing R at a suitable maximal ideal and completing (because R has a completely stable weak test element), and then the hypothesis of (b) holds. Thus, we assume the condition in (b) and suppose that $y = F(x) \in (I^{n+1})^*$ but that the coefficient r_μ of the monomial μ is not in I^*. Let J be the ideal generated by the monomials other than μ of degree n in the x's. Then $r_\mu \mu \in (I^{n+1})^* + J \subseteq (I^{n+1} + J)^*$, and so for some $c \in R^\circ$ we have that $cr_\mu^q \mu^q \in (I^{n+1} + J)^{[q]} = (I^{(n+1)})^{[q]} + J^{[q]}$ for all $q \gg 0$, which yields that $cr_\mu^q \in ((I^{(n+1)})^{[q]} + J^{[q]}) : \mu^q$. Formally, this is $I^{[q]}$ and so the right-hand side is contained in $(I^{[q]})^*$. This is sufficient for r_μ to be in I^* by Lemma (8.16) of [HH4]. □

(8.21) Remark. The requirement of a weak test element in the hypothesis for Theorem (8.20b) can be weakened: it suffices if there is an element $d \in R^\circ$ and a fixed power of p, say q', such that for any two ideals $\mathfrak{A}, \mathfrak{B}$ of R generated by monomials in the x's, if $\mathfrak{C}$ denotes the corresponding formal colon ideal then $c(\mathfrak{A} :_R \mathfrak{B})^{[q']} \subseteq \mathfrak{C}^{[q']}$. Note that then $c(\mathfrak{A} :_R \mathfrak{B})^{[qq']} = c((\mathfrak{A} :_R \mathfrak{B})^{[q]})^{[q']} \subseteq c(\mathfrak{A}^{[q]} :_R \mathfrak{B}^{[q]})^{[q']} \subseteq (\mathfrak{C}^{[q]})^{[q']} = \mathfrak{C}^{[qq']}$, since $\mathfrak{C}^{[q]}$ is the formal colon ideal for $\mathfrak{A}^{[q]}$, $\mathfrak{B}^{[q]}$. One may apply the argument in the proof of Lemma (8.16) of [HH4] without change. Roughly speaking, one only needs weak test elements for the kind of tight closure arising taking colons of ideals generated by monomials in parameters. Theorem (7.9) of [HH4] implies the existence of such a weak test element when the x's are permutable parameters in a locally equidimensional homomorphic image of a Cohen-Macaulay ring.

(8.22) Discussion. We conclude this section with a theorem that combines a result of Lipman and Sathaye [LS] with the theory of test elements developed in §§6 and 8 of [HH4] to give an easy method for getting specific information about what is in $\tau(R)$. We recall that if A is Noetherian domain with fraction field L and R is a module-finite extension ring of A, then R is *generically smooth over* A if $L \otimes_A R$ is a finite product of finite separable field extensions of L.

We shall suppose for simplicity that R is a domain containing A, module-finite over A, and generically smooth over A. The Jacobian ideal $\mathfrak{J}_{R/A} \subseteq R$ may then be defined as the 0th Fitting ideal for the module of differentials $\Omega_{R/A}$. We may calculate $\mathfrak{J}_{R/A}$ by choosing a presentation $R = T/I$, where $T = A[z_1, \cdots, z_n]$ is a polynomial ring over A. Then $\mathfrak{J}_{R/A}$ is the ideal of R generated by the images (under $T \twoheadrightarrow R$) of the Jacobian determinants $\partial(f_1, \cdots, f_n)/\partial(z_1, \cdots, z_n) \in T$ as $f_1, \cdots, f_n$ range through all choices of n-tuples of elements of I. If one chooses a specific finite set of generators of I, it suffices to let the f's be taken from among the n element subsets of these generators. The final result of this section is

(8.23) Theorem. *Let R be a domain module-finite and generically smooth over a regular Noetherian domain A of characteristic p. Then $\mathfrak{J}_{R/A} \cap A \subseteq \tau(R)$. In fact, every nonzero element of $\mathfrak{J}_{R/A} \cap A$ is a completely stable test element for R.*

Proof. By Theorem (6.13) of [HH4] generalized to the case of test elements for modules (see the discussion following Theorem (8.14) of [HH4]), an element c of A° will be a completely stable test element for R provided that $cR^{\infty} \subseteq A^{\infty}[R]$, where R^{∞} denotes $\bigcup_q R^{1/q}$ (the ring obtained from R by adjoining all qth roots of elements of R for all $q = p^e$). Thus, it suffices to show that $\mathfrak{J}_{R|A}$ multiplies R^{∞} into $A^{\infty}[R]$, and so it suffices to show that for all $q = p^e$, $\mathfrak{J}_{R/A}$ multiplies $R^{1/q}$ into $A^{1/q}[R]$, which by Lemma (6.4) of [HH4] may be identified with $A^{1/q} \otimes_A R$. Since the fraction field of R is separable over the fraction field of A, we know that $R^{1/q}$ has the same fraction field as

$A^{1/q} \otimes_A R \cong A^{1/q}[R]$, and so is contained in the normalization $\overline{S}$ of $S = A^{1/q} \otimes_A R$. Thus, it suffices to show that $\mathfrak{J}_{R/A}$ multiplies $\overline{S}$ into S. Let $B = A^{1/q}$, which is still regular and, hence, normal. If we write $R = A[z]/I$, where $z = z_1, \cdots, z_n$, then $S = B[z]/IB[z]$, and so $\mathfrak{J}_{S/B}$ is simply $\mathfrak{J}_{R/A}S$. Thus, it will suffice to show that $\mathfrak{J}_{S/B}$ is contained in the conductor of $\overline{S}$ into S. But this is immediate from Theorem 2, p. 200, of [LS] (it suffices to note that, in the notation of [LS], $1 \in \overline{S} : \overline{J}$). $\square$

(8.24) Remarks and an example. In trying to prove that $\tau(R)$ is large we are free to represent R as a finitely generated module over a regular ring A in many ways. Note that in looking for a multiple in A° of an element $r \in \mathfrak{J}_{R/A} - \{0\}$ we may always use the field norm of r (the characteristic polynomial of multiplication by r may be used to see that the field norm is a multiple of r (cf. (6.6) of [HH4])), but this is often inefficient.

For example, suppose that $R = K[z_0, \cdots, z_d]/(f)$ (or $K[[z]]/(f)$) where K is a field of characteristic p and $f = z_0^{a_0} + \cdots + z_d^{a_d}$ where every a_i is a positive integer not divisible by p. By viewing R as module-finite over the ring A_i obtained by adjoining all the z's except z_i to K and using the obvious presentation, we see that $z_i^{a_i - 1}$ (a_i is a unit) is in $\mathfrak{J}_{R/A_i}$. This element is not in A_i, but $z_i^{a_i}$ is in A_i, and so we see that $\tau(R) \supseteq (z_0^{a_0}, \cdots, z_d^{a_d})R$.

In general, this inequality is strict. For suppose that $d \geq 2$ and we let all the a_i be $d + 1$. Suppose that p is a prime that is congruent to 1 modulo $d + 1$. Then R is F-pure by Proposition (5.21c) on p. 157 of [HR2]. It follows from (8.13) that $\tau(R) \supseteq (z_0, \cdots, z_d)R = m$. Since m is a maximal ideal, and since R is not weakly F-regular (e.g., in the graded case $\mathbf{a}(R) = 0$, which shows the failure of weak F-regularity for the localization at m and for the completion as well) we must have that $\tau(R) = m$ here. Notice that Corollary (8.14) then applies.

Bibliography

[Ab1] I. Aberbach, *Finite phantom projective dimension and a phantom analogue of the Auslander-Buchsbaum theorem*, Thesis, Univ. of Michigan, Ann Arbor, 1990.

[Ab2] ——, *Finite phantom projective dimension*, Amer. J. Math. (to appear).

[Ab3] ——, *Tight closure in F-rational rings*, preprint.

[AHH] I. Aberbach, M. Hochster, and C. Huneke, *Localization of tight closure and modules of finite phantom projective dimension*, J. Reine Angew. Math. (Crelle's Journal) **434** (1993) 67–114.

[Bou] J.-F. Boutot, *Singularités rationelles et quotients par les groupes rédctifs*, Invent. Math. **88** (1987) 65–68.

[BrS] J. Briançon and H. Skoda, *Sur la clôture intégrale d'un idéal de germes de fonctions holomorphes en un point de* C^n, C. R. Acad. Sci. Paris Sér. A Math. **278** (1974) 949–951.

[BrV] W. Bruns and U. Vetter, *Determinantal rings*, Lecture Notes in Math., vol. 1327, Springer-Verlag, Berlin, 1988.

[DEP] C. De Concini, D. Eisenbud, and C. Procesi, *Hodge algebras*, Astérisque, no. 91, Math. Sci. France, Paris, 1982.

[Du] S. P. Dutta, *On the canonical element conjecture*, Trans. Amer. Math. Soc. **299** (1987) 803–811.

[EH] J. A. Eagon and M. Hochster, *R-sequences and indeterminates*, Quart. J. Math. Oxford Ser. (2) **25** (1974) 61–71.

[EvG1] E. G. Evans and P. Griffith, *The syzygy problem*, Ann. of Math. (2) **114** (1981) 323–333.

[EvG2] ——, *Syzygies*, London Math. Soc. Lecture Note Ser., vol. 106, Cambridge Univ. Press, Cambridge, 1985.

[EvG3] ——, *Order ideals*, Commutative Algebra, Math. Sci. Res. Inst. Publ., vol. 15, Springer-Verlag, New York-Berlin-Heidelberg, 1989, pp. 213–225.

[Fe1] R. Fedder, *F-purity and rational singularity*, Trans. Amer. Math. Soc. **278** (1983) 461–480.

[Fe2] ——, *F-purity and rational singularity in graded complete intersection rings*, Trans. Amer. Math. Soc. **301** (1987) 47–62.

[FeW] R. Fedder and K. Watanabe, *A characterization of F-regularity in terms of F-purity*, Commutative Algebra Math. Sci. Res. Inst. Publ., vol. 15, Springer-Verlag, New York-Berlin-Heidelberg, 1989, pp. 227–245.

[Fle] H. Flenner, *Rationale quasihomogene Singularitäten*, Arch. Math. **36** (1981) 35–44.

[Gla] D. J. Glassbrenner, *Invariant rings of group actions, determinantal rings, and tight closure*, Thesis, Univ. of Michigan, 1992.

[GrHa] A. Grothendieck (notes by R. Hartshorne), *Local cohomology*, Lecture Notes in Math., vol. 41, Springer-Verlag, Heidelberg, 1967.

[Ho1] M. Hochster, *Contracted ideals from integral extensions of regular rings*, Nagoya Math. J. **51** (1973) 25–43.

[Ho2] ——, *Topics in the homological theory of modules over commutative rings*, CBMS Regional Conf. Ser. in Math. no. 24, Amer. Math. Soc., Providence, RI, 1975.

[Ho3] ——, *Big Cohen-Macaulay modules and algebras and embeddability in rings of Witt vectors*, Proc. Queen's Univ. Commutative Algebra Conf., Queen's Papers in Pure and Applied Math., no. 42, Queen's Univ., Kingston, ON, 1975, pp. 106–195.

[Ho4] ——, *Cyclic purity versus purity in excellent Noetherian rings*, Trans. Amer. Math. Soc. **231** (1977) 463–488.

[Ho5] ——, *Some applications of the Frobenius in characteristic* 0, Bull. Amer. Math. Soc. **84** (1978) 886–912.

[Ho6] ——, *Cohen-Macaulay rings and modules*, Proc. Internat. Congr. Math. (Helsinki) Vol. I, Acad. Sci. Fenn., 1980, pp. 291–298.

[Ho7] ——, *Canonical elements in local cohomology modules and the direct summand conjecture*, J. Algebra **84** (1983) 503–553.

[Ho8] ——, *Grassmannians and their Schubert subvarieties are arithmetically Cohen-Macaulay*, J. Algebra **25** (1973) 40–57.

[Ho9] ——, *Solid closure*, Proc. Summer Research Conf. on Commutative Algebra (Mt. Holyoke College, July, 1992) (to appear).

[Ho10] ——, *Tight closure in equal characteristic, big Cohen-Macaulay algebras, and solid closure*, Proc. Summer Research Conf. on Commutative Algebra (Mt. Holyoke College, July, 1992) (to appear).

[HoE] M. Hochster and J. A. Eagon, *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math. **93** (1971) 1020–1058.

[HH1] M. Hochster and C. Huneke, *Tightly closed ideals*, Bull. Amer. Mat. Soc. (N.S.) **18** (1988) 45–48.

[HH2] ——, *Tight closure*, Commutative Algebra, Math. Sci. Res. Inst. Publ., vol. 15, Springer-Verlag, New York-Berlin-Heidelberg, 1989, pp. 305–324.

[HH3] ——, *Tight closure and strong F-regularity*, Mém. Soc. Math. France (N.S.), no. 38, Soc. Math. France, Montrouge, 1989, pp. 119–133.

[HH4] ——, *Tight closure, invariant theory, and the Briançon-Skoda theorem*, J. Amer. Math. Soc. **3** (1990) 31–116.

[HH5] ——, *Absolute integral closures are big Cohen-Macaulay algebras in characteristic* p, Bull. Amer. Math. Soc. (N.S.) **24** (1991) 137–143.

[HH6] ——, *Tight closure and elements of small order in integral extensions*, J. Pure Appl. Algebra **71** (1991) 233–247.

[HH7] ——, *Infinite integral extensions and big Cohen-Macaulay algebras*, Ann. of Math. (2) **135** (1992) 53–89.

[HH8] ——, *Phantom homology*, Mem. Amer. Math. Soc., No. 490 (1993) pp. 1–91.

[HH9] ——, *F-regularity, test elements, and smooth base change*, preprint.

[HH10] ——, *Tight closure in equal characteristic zero*, in preparation.

[HH11] ——, *Applications of the existence of big Cohen-Macaulay algebras*, preprint.

[HoRa] M. Hochster and L. J. Ratliff, Jr., *Five theorems on Macaulay rings*, Pacific J. Math. **44** (1973) 147–172.

[HR1] M. Hochster and J. L. Roberts, *Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay*, Adv. in Math. **13** (1974) 115–175.

[HR2] ——, *The purity of the Frobenius and local cohomology*, Adv. in Math. **21** (1976) 117–172.

[Hu1] C. Huneke, *Hilbert functions and symbolic powers*, Michigan Math. J. **34** (1987) 293–318.

[Hu2] ——, *An algebraist commuting in Berkeley*, Math. Intelligencer **11** (1989) 40–52.

[Hu3] ——, *Uniform bounds in Noetherian rings*, Invent. Math. **107** (1992) 203–223.

[Hu4] ——, *Absolute integral closures and big Cohen-Macaulay algebras*, Proc. Internat. Congr. Math. (Kyoto, 1990), Vol. I, Math. Soc. Japan, Springer-Verlag, New York-Berlin-Heidelberg, 1991, pp. 339–349.

[It1] S. Itoh, *Integral closures of ideals generated by regular sequences*, J. Algebra **117** (1988) 390–401.

[It2] ——, *Integral closures of ideals of the principal class*, Hiroshima Math. J. **17** (1987) 373–375.

[Ke] G. Kempf, *The Hochster-Roberts theorem of invariant theory*, Michigan Math. J. **26** (1979) 19–32.

[Lak] D. Laksov, *The arithmetic Cohen-Macaulay character of the Schubert schemes*, Acta Math. **129** (1972) 1–9.

[LS] J. Lipman and A. Sathaye, *Jacobian ideals and a theorem of Briançon-Skoda*, Michigan Math. J. **28** (1981) 199–222.

[LT] J. Lipman and B. Teissier, *Pseudo-rational local rings and a theorem of Briançon-Skoda about integral closures of ideals*, Michigan Math. J. **28** (1981) 97–116.

[Ma] F. Ma, *Splitting in integral extensions, Cohen-Macaulay modules and algebras*, J. Algebra **116** (1988) 176–195.

[MRo] J. Matijevic and P. Roberts, *A conjecture of Nagata on graded Cohen-Macaulay rings*, J. Math. Kyoto Univ. **14** (1974) 125–128.

[Mur] M. P. Murthy, *A note on factorial rings*, Arch. Math. (Basel) **15** (1964) 418–420.

[Mus] C. Musili, *Postulation formula for Schubert varieties*, J. Indian Math. Soc. **36** (1972) 143–171.

[PS1] C. Peskine and L. Szpiro, *Dimension projective finie et cohomologie locale*, Publ. Math. Inst. Hautes Études Sci. **42** (1973) 323–395.

[PS2] ——, *Syzygies et multiplicités*, C. R. Acad. Sci. Paris Sér. A Math. **278** (1974) 1421–1424.

[Re] D. Rees, *A note on analytically unramified local rings*, J. London Math. Soc. **36** (1961) 24–28.

[Ro1] P. Roberts, *Two applications on dualizing complexes over local rings*, Ann. Sci. École Norm. Sup. (4) **9** (1976) 103–106.

[Ro2] ——, *Cohen-Macaulay complexes and an analytic proof of the new intersection conjecture*, J. Algebra **66** (1980) 225–230.

[Ro3] ——, *The vanishing of intersection multiplicities of perfect complexes*, Bull. Amer. Math. Soc. (N.S.) **13** (1985) 127–130.

[Ro4] ——, *Le théorème d'intersection*, C. R. Acad. Sci. Paris Sér. I Math. **304** (1987) 177–180.

[Ro5] ——, *Intersection theorems*, Commutative Algebra, Math. Sci. Res. Inst. Publ., vol. 15, Springer-Verlag, New York-Berlin-Heidelberg, 1989, pp. 417–436.

[Sam] P. Samuel, *Lectures on Unique Factorization Domains*, Lecture Notes in Math., No. 30, Tata Institute, Bombay, 1964.

[Sk] H. Skoda, *Applications des techniques L^2 a la théorie des idéaux d'une algèbre de fonctions holomorphes avec poids*, Ann. Sci. École Norm. Sup. (4) **5** (1972) 545–579.

[Sm1] K. E. Smith, *Tight closure of parameter ideals and F-rationality*, Thesis, Univ. of Michigan, 1993.

[Sm2] ——, *Tight closure of parameter ideals*, Invent. Math. **115** (1994) 41–60.

[Sm3] ——, *F-rational rings have rational singularities*, preprint.

[Sw1] I. Swanson, *Joint reductions, tight closure, and the Briançon-Skoda theorem*, J. Algebra **147** (1992) 128–136.

[Sw2] ——, *Tight closure, joint reductions, and mixed multiplicities*, Thesis, Purdue Univ, 1992.

[Ve1] J. Velez, *Openness of the F-rational locus, smooth base change and Koh's conjecture*, Thesis, Univ. of Michigan, 1993, preprint, Journal of Algebra (to appear).

[W1] K.-i. Watanabe, *Rational singularities with k^*-action*, Commutative Algebra (Proc. of the Trento Conference), Lecture Notes in Pure and Appl. Math., vol. 84, Dekker, New York and Basel, 1983, pp. 339–351.

[W2] ——, *Study of F-purity in dimension two*, Algebraic Geometry and Commutative Algebra in honor of Masayoshi Nagata, vol. II, Kinokuniya, Tokyo, 1988, pp. 791–800.

[W3] ——, *F-regular and F-pure normal graded rings*, J. Pure Applied Algebra **71** (1991) 341–350.

[We] H. Weyl, *The classical groups*, Princeton Univ. Press, Princeton, NJ, 1946.

[Wil] L. Williams, *Uniform stability of kernels of Koszul cohomology indexed by the Frobenius endomorphism*, Thesis, Univ. of Michigan, 1992, preprint, Journal of Algebra (to appear).

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1003
E-mail address: hochster@math.lsu.umich.edu

Department of Mathematics, Purdue University, West Lafayette, IN 47907
E-mail address: huneke@math.purdue.edu