Math 615,Winter 2022 **Problem Set #1**
Due: Wednesday, February 2

In these problems, X, Y, Z are indeterminates and the corresponding lower case letters represent their images in a quotient. K is always a given field, and R is a given ring.

1. Let $R = K[X, Y, Z]/(X^3 + Y^3 + Z^3) =: K[x, y, z]$. Show that if K has characteristic p where $p \equiv 2$ modulo 3, then $z^2 \in (x, y)R^{1/p}$ (equivalently, $z^{2p} \in (x^p, y^p)R$). Hence, $R \subseteq R^{1/p}$ does not split. Thus, R is not a splinter.

2. Let $(R, \mathfrak{m}, K)$ be local (resp., an $\mathbb{N}$-graded ring with homogeneous maximal ideal $\mathfrak{m}$ and $R_0 = K$). By the *socle* in an R-module (resp., graded R-module) M we mean the submodule $\mathrm{Ann}_M \mathfrak{m}$. Show that if I is a $\mathfrak{m}$-primary ideal then I is tightly closed in R if and only if no element of R that represents an element of the socle in R/I is in the tight closure of I. (Suggestion: show that if an element of M is killed by a power of $\mathfrak{m}$, then it has multiple in the socle of M. Take $M = R/I$.)

3. Let R be a Cohen-Macaulaylocal ring. Show that the K-vector space dimension t of the socle of $R/\underline{x}$, where $\underline{x} = x_1, \ldots, x_d$ is a system of parameters, is independent of the choice of $\underline{x}$. (By a result in the notes, this reduces to the case where the two systems of parameters differ in only one element. By killing the parameters in the overlap, one may assume that R has dimension 1. If x, y are parameters, it suffices to compare the socles modulo x (and, hence, modulo y) with the socle modulo xy.) The integer t is called the *type* of R. Type one Cohen-Macaulaylocal rings are called *Gorenstein.*

4. Let R be a Gorenstein local ring (see the preceding problem) and let u generate the socle modulo the system of parameters $\underline{x}$. Then $(\underline{x})$ is tightly closed if and only u is not in the tight closure of $(\underline{x})$.

5. Suppose N is tightly closed in M and $f \in R$. Show that $N :_M f = m \in M : fm \in N$ is tightly closed in M.

6. Let R be a Noetherian domain of characteristic p and let M be an R-module (not necessarily finitely generated) on which we have a *Frobenius action*, i.e., a map $F : M \to M$ such that $F(rm) = r^p F(m)$ for all $m \in M$. Suppose $u \in M$ is fixed by F, i.e., $F(u) = u$, and we have an R-linear map $\theta : M \to R$ such that $\theta(u) = c \in R \setminus \{0\}$. Suppose that $I \subseteq J$ are ideals of R such that $IM = JM$. Prove that $J \subseteq I^*$.

Extra Credit 1. Let $(A, \mathfrak{m})$ by an Artin local ring containing a field K. By the structure theory of complete local rings, we may assume we have a composite isomorphism $K \subseteq A \twoheadrightarrow A/\mathfrak{m} = K$. Let $E = \mathrm{Hom}_(A, K)$. Show that we have a natural identification on A modules $\mathrm{Hom}_A(M, E) \cong \mathrm{Hom}_K(M, K)$ as contravariant functors of M (use the adjointness of tensor and Hom). Hence, $\mathrm{Hom}_A(\_\,, E)$ is an exact functor, i.e., E is an injective A-module. This functor preserves length. Show that every finitely generated A-module M is a submodule of a direct sum of copies of E, where the number of copies used is the dimension of the socle in M.

Extra Credit 2. In the situation of Extra Credit 1, let $M^\vee = \mathrm{Hom}_A(M, A) \cong \mathrm{Hom}_K(M, K)$. Note that $M \cong M^{\vee\vee}$. Show that for a finitely generated module M, the dimension of the socle in M (resp., $M^\vee$) is equal to the number of generators of $M^\vee$ (resp., M). Show that $A \cong A^\vee$ iff A is Gorenstein.