Math 615, Winter 2022 **Problem Set #2**

Due: Monday, February 21

1. (a) Let R be a domain and let C be an R-module such that every element of C is a torsion-element (i.e., killed by a nonzero element of R). Show that for all M and for all i, every element of $\mathrm{Tor}_i^R(M, C)$ is a torsion element. (You may assume that tensor and Tor_i commute with direct limits.)

(b) Let C be as in part (a), and suppose also that $0 \to A \to B \to C \to 0$ is a short exact sequence of R-modules such that $M \otimes_R A$ is torsion-free. Show that the map $M \otimes_R A \to M \otimes_R B$ is injective.

2. Let R be a subring of the polynomial ring $S = K[x_1, \ldots, x_n]$, where K is an infinite field of characteristic $p > 0$, that is generated by finitely many monomials. Let $I \subseteq R$ be generated by monomials. Assume that R has a nonzero test element (which is true).

(a) Explain very briefly why R must have a test element that is a monomial.

(b) Show that I^* is spanned by monomials, and that a monomial $\mu \in R$ is in I^* if and only if there exists a monomial ν in I and an integer $s > 0$ such that $(\mu/\nu)^s \in R$.

3. Let $R = K[[x, y]]/(x^2, xy)$. Let $\mathfrak{m} = (x, y)R$. Show that every minimal module of syzygies $\mathrm{syz}^n(K)$ of $K := R/\mathfrak{m}$ has the form $K^{\oplus a_n} \oplus \mathfrak{m}^{\oplus b_n}$, and determine recursions for a_n, b_n.

4. Let N be a submodule of a finitely generated module M over a Noetherian ring R of characteristic p. Let $q = p^e$. Show that $(N_M^*)^{[q]} \subseteq (N_M^{[q]})^*$.

5. The *Frobenius closure* I^{F} of I is the set $\{r \in R : \text{ for some } q = p^e, r^q \in I^{[q]}\}$. You may assume that I^{F} is an ideal containing I.

(a) Prove that $I^{\mathrm{F}} \subseteq I^*$.

(b) Assume that R is reduced. Show that $I = I^{\mathrm{F}}$ for every ideal $I \subseteq R$ iff every ideal of R^p is contracted from R, where $R^p = \{r^p : r \in R\}$.

6. Assume that R is F-split. Show that if $N \subseteq M$ are R-modules, and $u^q \in N^{[q]} \subseteq M^{[q]}$, then $u \in N$. Use this to prove that $\tau(R)$ is radical.

Extra Credit 3. Let $T := K[x_1, x_2]$, the polynomial ring in two variables over a field K of characteristic $p > 0$, and let $(R, \mathfrak{m})$ be the localization of $S := K[x_1^2, x_1x_2, x_2^2] \subseteq T$ at its homogeneous maximal ideal. S is spanned over K by all monomials of even degree. Determine $\lim_{q\to\infty} \ell(R/\mathfrak{m}^{[q]})/q^2$ (the Hilbert-Kunz multiplicity of R).

You may assume the following result. Let $x_1, \ldots, x_d$ be a regular sequence on M. Let $\mathcal{M}$ be the set of monomials in $x_1, \ldots, x_d$. Let I be an ideal generated by elements of $\mathcal{M}$ containing a power of every x_i. Let $\mu \subseteq \mathcal{M}$ be a monomial not a multiple of any generator of I such that $x_j\mu \in I$, $1 \leq j \leq d$. Then $(I + \mu)M/IM \cong M/(x_1, \ldots, x_d)M$.

Extra Credit 4. Let $(R, \mathfrak{m}, K)$ be a local ring of Krull dimension d with $\mathfrak{m} = (x_1, \ldots, x_d)$, and let M be an R-module of finite length $\ell_R(M)$. Suppose that $R \to S$ is local and flat (hence, faithfully flat). Show that $\ell_S(S \otimes M) = \ell_R(M)\ell_S(S/\mathfrak{m}S)$. Conclude that if R is regular, then $\ell_R\big(\mathcal{F}^e(M)\big) = q^d\ell(M)$, where $q = p^e$.