Math 615, Winter 2022 **Problem Set #3**
Due: Wednesday, March 16

1 Let R be a Noetherian ring of positive characteristic p.

(a) Show that for every ideal I of R, there exists a tightly closed ideal between I and $I :_R \tau(R)$.

(b) Show that if $\mathfrak{m}$ is a maximal ideal of R, $\tau(R)$ is $\mathfrak{m}$-primary, and I is an ideal of R such that $\mathfrak{m}$ is not an associated prime of I, then I is tightly closed.

2. Let $R = K[x^2, xy, y^2] \subseteq K[x, y]$, the polynomial ring in two variables over K, a field of characteristic $p > 0$. Let $\mathfrak{m} = (x^2, xy, y^2)$, the maximal ideal. Show that the map $\mathcal{F}^e_R(\mathfrak{m}) \twoheadrightarrow \mathfrak{m}^{[p^e]}_R \subseteq \mathcal{F}^e(R) \cong R$ (so that $\mathfrak{m}^{[p^e]}_R$ is the usual $\mathfrak{m}^{[p^e]}$) has a nonzero kernel. Give a presentation of this kernel when $e = 1$.

3. Let R be a domain with integral closure S that is module-finite over R. Assume that S is weakly F-regular. Let $H \subseteq R^{\oplus n} \subseteq S \otimes_R R^{\oplus n}$, which we may identify with $S^{\oplus n}$. Prove that $H^*_{R^{\oplus n}} = HS \cap R^n$, where HS is the submodule of $S^{\oplus n}$ generated by H.

4. Let $x_1, \, \dots \, , x_n, \, x'_n$ be arbitrary elements of an arbitrary ring R and let M be any R-module. Let $\underline{x} = x_1, \, \dots \, , x_n$, let $\underline{x}'' = x_1, \, \dots \, , x_{n-1}, x_n x'_n$, and let $\underline{x}' = x_1, \, \dots \, , x_{n-1}, x'_n$. Prove that there is a long exact sequence of Koszul homology:

$$\cdots \to H_i(\underline{x};\, M) \to H_i(\underline{x}'';\, M) \to H_i(\underline{x}';\, M) \to H_{i-1}(\underline{x};\, M) \to \cdots$$

[Suggestion; reduce to the case where $R = \mathbb{Z}[X_1, \, \dots \, , X_n, \, X'_N]$ is a polynomial ring.]

5. Over a local ring $(R, \, \mathfrak{m}, \, K)$, if M is a finitely generated R-module, prove that one has an isomorphism $\mathrm{Ext}^i_R(M, K) \cong \mathrm{Hom}_K\big(\mathrm{Tor}^R_i(M, K), K\big)$. (Hence, they are vector spaces of the same dimension.)

6. Let R be Noetherian, let M be finitely generated, and let $\underline{x} = x_1, \, \dots \, , x_n$ in R be such that (†) $\quad (x_1, \, \dots \, , x_n)R + \mathrm{Ann}_R M$ is primary to a maximal ideal $\mathfrak{m}$ of R.

(a) Show that the Koszul homology modules $H_i(\underline{x};\, M)$ have finite length.

(b) If (†) holds, define the *Koszul Euler characteristic of M with respect to $\underline{x}$*, denoted $\chi(\underline{x};\, M)$, as $\sum_{i=0}^{n} (-1)^i \ell\big(H_i(\underline{x};\, M)\big)$. Show that if M', M, and M'' all satisfy (†) and $0 \to M' \to M \to M'' \to 0$ is exact, then $\chi(\underline{x};\, M) = \chi(\underline{x};\, M') + \chi(\underline{x}; M")$.

(c) Show that if $x_1, \, \dots \, , x_n$ is any sequence of elements of $\mathfrak{m}$ and M is killed by a power of a maximal ideal $\mathfrak{m}$ (so that (†) is automatic), then $\chi(\underline{x};\, M) = 0$.

Extra Credit 5. Let $R := K[x^2, x^5] \subseteq K[x]$, the polynomial ring in x over an infinite field K of characteristic $p > 0$. Prove that the test ideal $\tau(R)$ is $(x^5, x^6)R$, and that $\tau(R) = \tau(R)^{\mathrm{F}}$, but $\tau(R)$ is not radical. (By a previous problem, $\tau(R)$ is generated by powers of x.)

Extra Credit 6. Let $(R, \mathfrak{m})$ be an Artin local ring and let $x, y \in \mathfrak{m}$. Prove that $H_1(x,\, y;\, R)$ needs at least two generators.