Math 615, Winter 2022 **Problem Set #4**
Due: Monday, April 4, 2022

1. Let R be a finitely generated $\mathbb{N}$-graded algebra over a field K, let $\mathfrak{m}$ be the homogenous maximal ideal, and let $F_1, \ldots, F_d$ be a sequence of homogeneous elements of positive degree. Show that the F_i are a regular sequence in R if and only if their images are a regular sequence in $R_\mathfrak{m}$. You may assume that the associated primes of a homogeneous ideal in an $\mathbb{N}$-graded Noetherian ring are homogeneous.

2. (a) Let M be a module over a local ring $(R, \mathfrak{m}, K)$ and let E be an injective hull of M. Prove that every element $u \in \mathrm{Ann}_E\mathfrak{m}$, the socle of E, is in the image of M in E.
(b) Consider a minimal injective resolution $E^\bullet$ of a module M over a local ring $(R, \mathfrak{m}, K)$. Prove that the maps in the complex $\mathrm{Hom}_R(K, E^\bullet)$ are all 0.

3. Let $X := (x_{ij})$ be a 3×2 matrix of indeterminates over a field K. Let Δ_i denote $(-1)^{i-1}$ times the size 2 determinant obtained by deleting the ith row of X. Let Δ be the matrix $(\Delta_1\ \Delta_2\ \Delta_3)$. You may assume that

$$0 \to S^2 \xrightarrow{\ X\ } S^3 \xrightarrow{\ \Delta\ } S \to 0$$

is a free resolution of $R := S/I$. Prove that $\mathrm{Ext}^2_S(R, S) \cong (x_{1,1}, x_{1,2})R$.

4. Let X_{ij} be an $n \times n$ matrix of indeterminates over a field K, and let $R = K[X] := K[X_{ij} : 1 \le i, j \le n]$. Prove that the coefficients of the characteristic polynomial of X form a regular sequence in R.

5. Let $R = K[x^4, x^3y, x^2y^2, y^4] \subseteq K[x, y]$ and let S be the localization of R at its homogeneous maximal ideal $\mathfrak{m}$. Determine whether S is Cohen-Macaulay.

6. Let S be the ring generated by all forms of degree d in the polynomial ring $T = K[x_1, \ldots, x_n]$. Note that $S \to T$ splits as a map of S-modules (the complement is spanned over K by all forms of degree not divisible by d). Show that $x_1^d, \ldots, x_n^d$ is a regular sequence in S and, hence, in $S_\mathfrak{m}$, where $\mathfrak{m}$ is the homogeneous maximal ideal of S. Thus, $S_\mathfrak{m}$ is Cohen-Macaulay. Determine the type of $S_\mathfrak{m}$ as a function of d and n.

Extra Credit 7. Let $(R, \mathfrak{m}, K)$ be a local ring, $I \subseteq \mathfrak{m}$ an ideal, and let E be an injective hull of $K = R/\mathfrak{m}$ over R. Show that $\mathrm{Ann}_E I$ is an injective hull of the residue class field K of R/I over R/I (note that $(R/I)/(\mathfrak{m}/I) \cong R/\mathfrak{m} = K$).

Extra Credit 8. Let $X = (X_{ij})$ be an $r \times s$ matrix of indeterminates over an algebraically closed field K, $1 \le r \le s$. Let P be the ideal generated by the size r minors of X in $S = K[X_{ij} : 1 \le i \le r, 1 \le j \le s]$. You may assume that S/P is a Cohen-Macaulay domain of dimension $rs - s + r - 1$. Let R be the localization of S/P at its homogeneous maximal ideal. Consider the $s-r+1$ "diagonals" $X_{1,t}, X_{2,t+1}, \ldots, X_{r,t+r-1}$, where $1 \le t \le s-r+1$. Show that the elements below the first diagonal, the differences of the consecutive elements on these diagonals, and the elements above the last diagonal together form a system of parameters for S, and show that the quotient by the ideal generated by these elements is isomorphic with $K[x_1, \ldots, x_{s-r+1}]/(x_1, \ldots, x_{s-r+1})^r$. What is the type of R?