1. Since the $F_1, \ldots, F_d \in \mathfrak{m}$, they generated a proper ideal in both rings. We have already seen the flat base change preserves the property of being a nonzerodivisor and commutes with taking quotients. It follows that if $F_1, \ldots, F_d$ is a regular sequence R, it remains one in $R_\mathfrak{m}$. For the converse, note that since the associated primes of each ideal $J_i = (F_1, \ldots, F_i)R$, $0 \le i \le d-1$, are ntahomogenous, and so contained in $\mathfrak{m}$, no element of $W := R \setminus \mathfrak{m}$ is a zerodivisor on R/J_i, we have that R/J_i injectd into $(R/J_i)_\mathfrak{m} \cong R_\mathfrak{m}/J_iR\mathfrak{m}$. It follows that for $0 \le i \le d-1$, if the image of F_i is a nonzerodivisor on $R_\mathfrak{m}/J_iR_\mathfrak{m}$, it is a nonzerodivisor in R/J_i.

2. (a) $Ru \cong Ku$, and u has a nonzero multiple in M. This multiple must have the form cu, where $c \in K$ is nonzero, and so $u \in M$.

(b) $\mathrm{Hom}_R(K, E_i)$ is the socle S_i in E_i. Since this is aa minimal injective resolution, E_i is the injective hull of the image B_i of E_{i-1} in E_i. By part (a), each element of S_i is in B_i, and so maps to 0 in E_{i+1}.

3. We compute the Ext module by applying $\mathrm{Hom}_S(\_\,, S)$ to the resolution and taking cohomology. When we do this, the arrows are reversed, and the matrices with respect to the dual bases for the dual free modules are the transposes. It follows that $\mathrm{Ext}^2_S(R, S)$ is the cokernel of the map $S^3 \to S^2$ with matrix $X\mathrm{tr}$. We know *a priori* that this is an R-module, since $\mathrm{Ann}_S R$ kills the Ext module. Map S^2 onto $(x_{11}, x_{12})R$ by $(f, g) \mapsto x_{12}f - x_{11}g$. Then it suffices to show that the kernel of this map is spanned by the columns of $X\mathrm{tr}$, i.e., by the rows of X. The columns are in the kernel, since the size 2 minors Δ_i of X vanish in R. Working over S, an element (f, g) of the kernel corresponds to a solution of $x_{12}f - x_{11}g = a\Delta_1 + b\Delta_2 + c\Delta_3$. Note that all relations with $f, g \in (\Delta_1, \Delta_2, \Delta_3) := P$ are obtainable from the three rows, because we know that the the cokernel of $X\mathrm{tr}$, which is $\mathrm{Ext}^2_S(R, S)$, is killed by $\mathrm{Ann}_S R = (\Delta_1, \Delta_2, \Delta_3)$. We then have that $x_{12}f \in (x_{11}, \Delta_i : 1 \le i \le 3) \subseteq (x_{11}, x_{21}, x_{31})S$. It follows that $f \in (x_{11}, x_{21}, x_{31})S$. Hence by subtracting an S-linear combination of the rows of X from (f, g), we get a new relation in which $f = 0$. It will then suffice to show that $g \in (\Delta_1, \Delta_2, \Delta_3)$. In this case, we have $-x_{11}g = a\Delta_1 + b\Delta_2 + c\Delta_3 \in (x_{11}, x_{12}x_{21}, x_{12}x_{31}, \Delta_1)$. The result now follows if we know that P is prime. In fact, P is the kernel Q of the map from $K[X]$ to the Segre product $T := K[s_1, s_2, s_3]\#_K K[t_1, t_2]$ such that $x_{ij} \mapsto s_it_j$. In the latter ring, each monomial such that the degree in the s_i is equal to the degree in the t_i can be written uniquely as a product of s_it_j so as to use as many as possible terms of the form s_1t_1, then s_1t_2, then s_2t_2, then s_2t_2, then s_3t_1, then s_3t_2. The relations given by the minors enable to one to prove congruent mod P any two monomials α, β in the x_{ij} with the same image μ in T. Use induction on the degrees of α, β. Let i be smallest such that s_i occurs in μ and j be smallest such that t_j occurs in μ. If each of α, β is congruent to a monomial involving x_{ij}, we can factor x_{ij} out and use the induction hypothesis. Suppose x_{ij} does not occur, say, in α. Then $x_{ij'}$ occurs, where $1' = 2$ and $2' = 1$. Then $j' = 1$ or we could have made a smaller choice of j. Also, x_{i*j} must occur for some $i*$, and $i* > i$ or we could have made a smaller choice of i, But $x_{i*j}x_{ij'} \equiv x_{ij}x_{i*j'}$, so α is congruent to a monomial involving x_{ij} mod the minors generating P. The same argument can be applied to β. $\square$

4. Since we are in an $\mathbb{N}$-graded ring and working with homogeneous elements of positive degree, it suffices to show that the off-diagonal entries of the matrix and the coefficients (other than 1) form a regular sequence, for that regular sequence will be permutable and we can place the coefficients first. It is clear that the off-diagonal variables form a regular sequence. Once we have killed these, the matrix becomes diagonal with entries $x_i := x_{i,i}$., and the characteristic polynomial become the coefficiens of $(z-x_1)\cdots(z-x_n) = z^n + c_1z^{n-1} + \cdots + c_n$. Any value of z is a root of this polynomial satisfies the condition that $z^n \in (c_1, \ldots, c_n)$. Since each x_i is a root of this polynomial, it follows that each x_i has its nth power in the ideal generated by the coefficients. Hence, the coefficients generate an ideal whose radical in $K[x_1, \ldots, x_n]_(x_1, \ldots, x_n)$ is the maximal ideal. Thus, the coefficients are a system of parameters in the regular (hence, Cohen-Macaulay) ring $K[x_1, \ldots, x_n]$. Thus, they are a regular sequence in this localized ring. By Problem **1.**, they form a regular sequence in $K[x_1, \ldots, x_n]$. $\square$

5. The ring has dimension 2, since its integral extension $K[x, y]$ does, and the maximal ideal is nilpotent modulo (x^4, y^4), so x^4, y^4 will be a system of parameters in $R_\mathfrak{m}$. It is easy to show by induction on n that the monomials in the ring in degree $4n$, for $n \geq 1$, are all monomials of degree $4n$ except $x^{4n-1}y$. It is clear that one cannot obtain any monomial linear in y, and one does get all the others: once $n \geq 2$ if the exponent on x is at least 4 one can factor out x^4. If the exponent i on x is ≤ 3, the exponent on y is at least $4n - 3$ and one may factor out y^4 except when $n = 2$, but when $i = 3$, $n = 2$. But $x^3y^5 = (x^2y^2)(xy^3)$. If there is a relation $fx^4 = gy^4$, g is divisible by x^4 in the polynomial ring, and g/x^4 cannot have a term that is linear in y unless g does, so $g \in x^4R$. Thus x^4, y^4 is a regular sequence in R, and so in $R_\mathfrak{m}$. Hence, $R_\mathfrak{m}$ is Cohen-Macaulay. $\square$

6. Elements form a possibly improper regular sequence on a direct sum if and only if they form a form a possibly improper regular sequence in each nontrivial summand. Hence, $x_1^d, \ldots, x_n^d$ is a possibly improper regular sequence on S. Since these elements generate a proper ideal, they form a regular sequence. This remains true in $S_\mathfrak{m}$, where they are a system of parameters. Hence, $S_\mathfrak{m}$ is Cohen-Macaaulay. It remains to determine the dimension of the socle in $S/(x_1^d, \ldots, x_n^d)$. (This ring is already local.) This will be generated by all monomials $x_1^{d-1-a_1} \cdots x_n^{d-1-a_n}$ where $0 \leq a_i \leq d - 1$ ($a_i \geq 0$ guarantees that the monomial is not divisible by x_i^d) such that $\sum_{i=1}^n (a_i + 1)$ is a multiple of d (so that the monomial is in S) and such that for any choice of $x_1^{b_1} \cdots x_n^{b_n}$ with the $b_i \in \mathbb{N}$ such that $\sum_i b_i = d$, one has that at least one $a_i - 1 + b_i \geq d$. This precisely forces $\sum_{i=1}^n a_i < d$. (Otherwise one may choose values for $b_i \leq a_i$ whose sum is d, and $x_1^{b_1} \ldots x_n^{b_n}$ will not multiply $x_1 d - 1 - a_1 \cdots x_n^{d-1-a_n}$ into $(x_1^d, \ldots, x_n^d)$. Hence, we wish to count the n-tuples of nonnegative integers in with sum $s < d$ such that $s - n$ is divisible by d. Thus, the possibilities for s are the integers $dk - n$ with $0 \leq dk - n < d$, i.e., $n/d \leq k < (n/d) + 1$. Thus, $k = \lceil \frac{n}{d} \rceil$, $s = dk - n$, and we must count the number of choices of $a_1, \ldots, a_n \in \mathbb{N}$ with $\sum_{i=1}^n a_i = s$. Use consecutive dots to represent values of the a_i separated by $n - 1$ slashes. The number of choices for the a_i is the number of strings consisting of $s + n - 1 = dk - n + n - 1 = dk - 1$ elements with s dots and $n - 1$ slashes. These are determined by the placement of the slashes. Hence, the type is $\binom{\lceil \frac{n}{d} \rceil d - 1}{n - 1}$.

EC7. The injective hull of R/I is aa maximal essential extension of K as an R/I-module, and one may take a maximal extension of this as an R-module, which will also be a maximal essential extension of K. This gives an injection of $E_{R/I}(K)$ into $E_R(K)$. It is contained in the annihilator M of I in $E_R(K)$. But M is an (R/I)-module, and is an essential extension of K. Since $E_{R/I}(K) \subseteq M$ and $E_{R/I}(K)$ is a maximal essential extension, we must have

$E_{R/I}(K) = M$. $\square$

EC8. The number of elements in the proposed system of parameters is corrected, so it suffices to show that the the homogeneous maximal ideal is nilpotent mod these. If we renumber the first row by omitting the subscript 1, then after killing prescribed elements the matrix has 0 below the first diagonal, x_1 on the first diagonal, x_2 on the second diagonal, as so forth. We shall prove by induction on r and s that the ideal of minors is $(x_1, \dots, x_{s-r+1})^r$ power. The base cases are easy to check. For example, one of them is when $s = r$. In that case, the matrix is diagonal with x_1 all along the diagonal.

We shall also use induction on k to prove that we get all monomials that are divisible by any of $x_1, \dots, x_k$. From the size r minors that involve the first column and $r-1$ later columns, we get the product of x_1 and each minor of a similar matrix with r, s replace by $r-1, s-1$. By the induction hypothesis, we get all monomials of degree r that are divisible by x_1. Now consider the matrix formed from by omitting the first column. Modulo x_1, this is an $r \times (s-1)$ matrix of the same form, except that the variables on the diagonals are $x_2, \dots, x_{s-r+1}$. By the induction hypothesis, modulo (x_1) the minors generate the ideal $(x_2, \dots, x_{s-r+1})^r$. If we do not kill x_1, from the minors of this matrix we get, for every monomial μ of degree r in the variables $x_2, \dots, x_r$, an element of the form $\mu + x_1\delta$, where δ is a homogeneous polynomial of degree $r-1$. Since we already know that all degree r polynomials that are multiples of x_1 are in the ideal, $x_1\delta$ is in the ideal, and it follows that μ is in the ideal.

The type is the number of monomials of degree $r-1$ in $s-r+1$ variables, which is $\binom{s-r+r-1}{s-r} = \binom{s-1}{s-r} = \binom{s-1}{r-1}$.