**1.** Suppose $r \in R$ and $r \in (IS)^*$, so that for some $c \in S \setminus \{0\}$, $cr^q \in (IS)^{[q]} = I^{[q]}S$ for all $q \gg 1$. $c$ has a nonzero multiple in $R$, so we may assume $c \in R$. As shown in class, there is an $R$-linear map $\theta : S \to R$ such that $\theta(1) = d \neq 0$. Applying $\theta$ yields $cr^q\theta(1) \in I^{[q]}$ for all $q \gg 1$, and so $(cd)r^q \in I^{[q]}$ for all $q \gg 0$. $\square$

**2.** (a) Let $\mathfrak{m}_n$ be the ideal spanned by forms of degree $\geq n$. For a standard grading, $\mathfrak{m}_n = \mathfrak{m}^n$, but in all cases the $\mathfrak{m}_n$ are cofinal with the powers of $\mathfrak{m}$ and are $\mathfrak{m}$-primary. Since $E := E_R(K)$ is an essential extension of $K$, every element is killed by $\mathfrak{m}_n$ for some $n \gg 0$, and so $E = \bigcup_n \mathrm{Ann}_E \mathfrak{m}_n$. Since $R/\mathfrak{m}_n$ is Artin, we already know from class results that $\mathrm{Ann}_E \mathfrak{m}_n \cong E_{R/\mathfrak{m}_n}(K) \cong \mathrm{Hom}_K(R/\mathfrak{m}_n, K)$, and $\mathrm{Hom}_K(R/\mathfrak{m}_{n+1}, K) \cong \bigoplus_{t=0}^{n} \mathrm{Hom}_K([R]_t, K)$. It follows that $E = \bigoplus_{t=0}^{\infty} \mathrm{Hom}_K([R]_t, K)$. Note that the $R$-module structure on the space of linear functionals that kill some $\mathfrak{m}_n$ is given by $(r\theta)(f) = \theta(rf)$.

(b) If $b_1, \ldots, b_s$ is a basis for a finite-dimensional $K$-vector space $V$, let $b_1^\vee, \ldots, b_s^\vee$ denote the dual basis for the dual of $V$ into $K$, so that the functional $b_i^\vee$ is 1 on $b_i$ and 0 on $b_j$ for $j \neq i$. Grade the Segre product so that the typical element $\mu = r^a s^b t^c u^d$, where $a+b = c+d$, has degree $a+b$. Let $\mathcal{M}$ be the set of monomials in $R$. Then we may use the union of all the dual bases of the graded components of $R$ to give a basis $\{\mu^\vee : \mu \in \mathcal{M}\}$, where $\mathcal{M}$ is the set of all monomials in $R$, for the injective hull $E := E_R(K)$, by part (a). Let $\lambda := rstu$. There is a bijection $\theta$ of this dual basis with the strictly negative monomials specified in the problem given by $\theta : \mu^\vee \mapsto (\lambda\mu)^{-1}$. This extends to a $K$-vector space isomorphism of $E$ with $E_0$. To check that this is an $R$-module isomorphism one needs to check that for all $\nu \in \mathcal{M}$, $(*)\quad \nu\theta(\mu\vee) = \theta(\nu\mu\vee)$. Let $\overline{\mu/\nu}$ denote $\mu/\nu$ if $\nu|\mu$ and 0 otherwise. It is straightforward to check $\nu\mu^\vee = (\overline{\mu/\nu})^\vee$. Also, $\nu|\mu$ iff $\nu/(\lambda\mu)$ is striclty negative. It follows that each side of $(*)$ is nonzero if and only if $\nu$ divides $\mu$, and then both sides are $\nu/(\lambda\mu)$. $\square$

**3.** (a) If $u \in M \setminus \{0\}$ we have $Ru \twoheadrightarrow Ru/\mathfrak{m}(Ru) \cong K$, and so a nonzero composite map $f_0 : Ru \twoheadrightarrow K \hookrightarrow E$. Since $E$ is injective, $f_0$ extends to a nonzero map $f : M \to E$. $\square$

(b) If $u \in M \setminus \{0\}$, the map $f : M \to E$ constructed in part (a) is nonzero on $u$, and then the image $T_u$ of $u$ in $\mathrm{Hom}_R\big(\mathrm{Hom}_R(M, E), E\big)$ is nonzero, since $T_u(f) = f(u) \neq 0$. $\square$

**4.** (a) Let $x_1, \ldots, x_d$ be a system of parameters in $R$. By flatness, this is a regular sequence on $M$ (note that $(x_1, \ldots, x_d)S \subseteq \mathfrak{n}$ and $M/\mathfrak{n}M \neq 0$, by Nakayama's lemma). Since $\mathrm{Rad}\,(x_1, \ldots, x_d) = \mathfrak{m}$ and $\mathrm{Rad}\,(\mathfrak{m}S) = \mathfrak{n}$, it follows that $\mathrm{Rad}\,\big((x_1, \ldots, x_d)S\big) = \mathfrak{n}$, and so the images of $x_1, \ldots, x_d$ form a sequence of elements in $S$ such that $M/(x_1, \ldots, x_d)M$ is killed by a power of $\mathfrak{n}$. Since this sequence is a regular sequence on $M$, it follows that $\dim(M) = d$ and $M$ is Cohen-Macaulay. $\square$

(b) Let $f_1, \ldots, f_k$ generate $\mathfrak{m}$. We have a map $R \to R^k$ such that $1 \mapsto (f_1, \ldots, f_k)$ whose kernel is $\mathrm{Ann}_R\mathfrak{m}$. Since $0 \to \mathrm{Ann}_R\mathfrak{m} \to R \to R^k$ is exact and $M$ is $R$-flat, we have also that $0 \to (\mathrm{Ann}_R\mathfrak{m}) \otimes_R M \to M \to M^k$ is exact, where the rightmost map $\alpha$ is $u \mapsto (f_1u, \ldots, f_ku)$. $\mathrm{Ker}\,(\alpha) = \mathrm{Ann}_M\mathfrak{m}$, but from the exactness of $(*)$ is also the image (under an injection) of $(\mathrm{Ann}_R\mathfrak{m}) \otimes_R M$, which which may be identified with $(\mathrm{Ann}_R\mathfrak{m})M$. Then $\mathrm{Ann}_M\mathfrak{n} \subseteq \mathrm{Ann}_M\mathfrak{m}$, and is the same as $\mathrm{Ann}_{(\mathrm{Ann}_R\mathfrak{m})\otimes_R M}\mathfrak{n}$. Now $\mathrm{Ann}_R\mathfrak{m}$ is a $K$-vector space whose dimension is the type $t$ of $R$ and so is $\cong K^t$. Then $\mathrm{Ann}_M\mathfrak{n} \cong \mathrm{Ann}_{K^t \otimes_R M}\mathfrak{n} \cong \mathrm{Ann}_{(M/\mathfrak{m}M)^{\oplus t}}\mathfrak{n} \cong (\mathrm{Ann}_{M/\mathfrak{m}M}\mathfrak{n})^{\oplus t}$, and the dimension over $L$, which is the type of $M$ over $S$, is $t$ times the dimension over $L$ of $\mathrm{Ann}_{M/\mathfrak{m}M}\mathfrak{n}$, which is $t$ times the type of $M/\mathfrak{m}M$. $\square$

**5.** (a) Consider a maximal ideal $\mathfrak{m}'$ of $R^\Gamma$ and the corresponding maximal ideal $\mathfrak{m}$ of $R$. Then $R_m \to R\Gamma_{m'} = R^\Gamma_m$ is flat local, $R_\mathfrak{m}$ is Cohen-Macaulay, and the fiber is $(R/\mathfrak{m})^\Gamma_\mathfrak{m} \cong (R/\mathfrak{m})^\Gamma$, which is purely inseparable over the field $(R/\mathfrak{m})$, and so is zero-dimensional. By part (a) of Problem **4.**, $R^\Gamma_{m'}$ is Cohen-Macaulay. Since $m'$ is arbitrary, $R^\Gamma$ is Cohen-Macaulay. $\square$

(b) It was show in class that when $R$ is regular, so is $R^\Gamma$ for all sufficiently small $\Gamma \ll \Lambda$. When $R$ is not regular, by a class result if $V(J)$ is the singular locus in $R$, then for all sufficiently small $\Gamma \ll \Lambda$, $V(JR^\Gamma)$ is the singular locus in $R^\Gamma$. if $R$ is normal, we can choose a regular sequence of length two (or more) in $J$. Since $R \to R\Gamma$ is flat, the image of these elements is a regular sequence in $JR\Gamma$.

**6.** We may replace $R$ by $\widehat{R}$ without loss of generality, and so assume that $R$ is complete. When $E_i \subseteq E_j \subseteq E$ where $E_i \subset E_j$ is a proper inclusion, we may apply $\mathrm{Hom}_R(\_\,, E)$ to obtain surjections $R \cong \mathrm{Hom}_R(E, E) \to \mathrm{Hom}_R(E_j, E) \to \mathrm{Hom}_R(E_i, E)$ where, since $\mathrm{Hom}_R(\_\,, E)$ is faithfully exact by Problem **3.**(a), we have that the rightmost map is a proper surjection. Thus, dualizing produces $R \to R/J_j \to R/J_i$ where the second map is a proper surjection: this means $J_i$ is strictly larger than $J_j$. It will complete all parts of the problem if we show that $J_i = \mathrm{Ann}_R E_i =: \mathfrak{A}_i$. Since $R/J_i \cong \mathrm{Hom}_R(E_i, E)$, we have that $\mathfrak{A}_i \subseteq J_i$. But we also know that $E_i \to \mathrm{Hom}_R\big(\mathrm{Hom}_R(E_i, E), E\big)$ is injective by problem **3.**(b), and the latter module is killed by $J_i$ since $\mathrm{Hom}_R(E_i, E) \cong R/J_i$, which shows that $J_i \subseteq \mathfrak{A}_i$. $\square$ (In fact, over a complete local ring, if $M$ has ACC or DCC, then $\mathrm{Hom}_R(M, E)$ ACC or DCC, respectively, and $M \to \mathrm{Hom}_R\big(\mathrm{Hom}_R(M, E), E\big)$ is an isomorphism.)

**EC9.** We have $\mathrm{pd}_R M = \mathrm{ht}\,(I) = n - d$. Let $0 \to R^{b_h} \to \cdots \to R^{b_0} \to 0$ be a minimal free resolution of $M$, where $d_i : R^{b_i} \to R^{b_{i-1}}$ and $\mathrm{Coker}\,(d_1) \cong M$. When we apply $\_^* = \mathrm{Hom}_R(\_\,, R)$ (*not* tight closure!), we get: $(*)\quad 0 \to (R^{b_0})^* \to \cdots \to (R^{b_h})^* \to 0$, and the cohomology is $\mathrm{Ext}^\bullet_R(M, R)$. Here, the $(R^{b_i})^*$ are free, and the matrix of $d_i^*$ is the transpose of the matrix of $d_i$. Since the depth of $R$ on $\mathrm{Ann}_R M = I$ is $\mathrm{ht}\,(I)$, which is $h$, the $j$ th cohomology of the complex $(*)$ is $\mathrm{Ext}^j_R(M, R) = 0$ except when $j = h$. Hence, the complex $(*)$ is a minimal (entries of the matrices are still in $\mathfrak{m}$) free resolution of the $\mathrm{Coker}\,(d_h^*) = \mathrm{Ext}^h_R(M, R) = M^\vee$. Moreover, if use this resolution to compute $\mathrm{Ext}^h(M^\vee, R)$, because $\_^{**}$ is naturally isomorphic to the identity functor, we come back to the original complex, which shows that $\mathrm{Ext}^h_R(M^\vee, R) \cong M$. Clearly, $I$ kills $M^\vee$ and the annihilator of $M^\vee$ kills $\mathrm{Ext}^h_R(M^\vee, R)$. Since the annihilator of each module kills the other, each annihilator is contained in the other, and they are equal. Thus, $\dim(M^\vee) = n - h = d$. By the Auslander-Buchsbaum theorem, the depth of $M^\vee$ on $\mathfrak{m}$ is $n - \mathrm{pd}_R(M^\vee) = n - h = d$. Hence, $M^\vee$ is Cohen-Macaulay. Exactness of $\_^\vee$ now follows from the long exact sequence for Ext along with the fact that for Cohen-Macaulay modules of dimension $d$, the only non-vanishing $\mathrm{Ext}^j_R(M, R)$ occurs when $j = n - d$. When $x$ is a nonzerodivisor on $M$, $0 \to M \to M \to M/xM \to 0$ is exact, yield a long exact sequence for $\mathrm{Ext}_R(\_\,, R)$ with just three consecutive nonzero terms, and then $(M/xM)^\vee \cong M^\vee/xM^\vee$ is immediate. When we localize at a prime $P \supseteq I$, $R_P$ is regular, $M_P$ is Cohen-Macaulay, the new annihilator is $I_P$, which has height $h$ since all minimal primes of $I$ have height $h$, and so $M_P^\vee \cong \mathrm{Ext}^h_{R_P}(M_P, R_P) \cong \mathrm{Ext}^h_R(M, R)_P = (M^\vee)_P$. $\square$

**EC10.** (a) Use the Koszul complex of a regular system of parameters to calculate Ext.

(b) Fom the exactness of $\_^\vee$, if one has a short exact sequence of finite length modules

$0 \to A \to M \to B \to 0$, then $\ell(M^\vee) = \ell(A^\vee) + \ell(C^\vee)$. The result is immediate from (a) by induction on $\ell(M)$.

(c) The type of $M$ and the least number of generators of $M^\vee$ don't change when one passes from $M$ to $M/xM$, where $x$ is a nonzerodivisor on $M$, since $(M/xM)^\vee = M^\vee/xM^\vee$. Iterated application of this fact reduces the problem to the case where $M$ has finite length. The result then says that the socle in $M$ has the same dimension as the least number of generators of $M^\vee$. With notation as in the first part of Problem **4.**(b) we have that $0 \to \mathrm{Ann}_M\mathfrak{m} \to M \to M^k$ is exact, and thus, applying $^\vee$, we have that the sequence ${M^\vee}^k \to M^\vee \to (\mathrm{Ann}_M\mathfrak{m})^\vee \to 0$ is exact, where the leftmost map takes $(u_1, \ldots, u_k) \mapsto \sum_{i=1}^k f_i u_i$. Thus, the cokernel of the map is also $M^\vee/\mathfrak{m}M^\vee$, and the least number of generators of $M^\vee$, by Nakayma's lemma, is the length (the same as vector space dimension) of $M^\vee/\mathfrak{m}M^\vee$. This is the same as the length of $(\mathrm{Ann}_M\mathfrak{m})^\vee$, which by part (b) is the same as the length (or vector space dimension) of $\mathrm{Ann}_M\mathfrak{m}$, the socle of $M$. $\square$