Math 711: Lecture of September 8, 2004

We shall first prove Zariski's Main Theorem, and then define and analyze the structure of smooth, étale, and unramified homomorphisms. We shall discuss Henselization. Another segment of the course will return to the Serre theory of intersection multiplicities which was introduced in Math 615 last semester. Here, we shall give a more geometric notion and then show that it turns out to be the same as Serre's notion when it is defined. If time permits we will treat relatively recent results of Gabber on non-negativity of Serre multiplicities. These results have greater geometric prerequisites than the rest of the course.

Throughout these lectures, unless otherwise indicated, all rings are commutative, associative rings with multiplicative identity and ring homomorphisms preserve the identity. Modules are assumed to be unital.

Let $g : R \to S$ be a ring homomorphism and let $P \in \mathrm{Spec}\,(R)$ be a prime ideal of R. We want to recall the notion of the (scheme-theoretic) fiber of g over P. Let $\kappa_P = R_P/PR_P$, which is isomorphic with the fraction field of R/P, $\mathrm{frac}\,(R/P)$. Then the *fiber* of g over P is the S-algebra $\kappa_P \otimes_R S \cong (R-P)^{-1}(S/PS)$. The natural homomorphism $S \to \kappa_P \otimes_R S$ gives a map $\mathrm{Spec}\,(\kappa_P \otimes_R S) \to \mathrm{Spec}\,(S)$ which is a bijection of the former with the set of prime ideals of S containing PS and disjoint from the image of $R - P$ in S. This is precisely the set of prime ideals of S lying over (i.e., contracting to) P, and may also be described, as a set, as the set-theoretic fiber of the map $\mathrm{Spec}\,(f) : \mathrm{Spec}\,(S) \to \mathrm{Spec}\,(R)$ over P, i.e., as $\bigl(\mathrm{Spec}\,(f)\bigr)^{-1}(\{P\})$. It is easy to verify that the map is a homeomorphism of $\mathrm{Spec}\,(\kappa_P \otimes_R S)$ with this set-theoretic fiber over P with the inherited topology from $\mathrm{Spec}\,(S)$. This enables us to think of the set-theoretic fiber as having the structure of Spec of a ring.

With this notation, a prime Q of S that lies over P in R is called *isolated in its fiber* or *isolated in the fiber over P* if it is both maximal and minimal among primes lying over P. In particular, if Q is the unique prime lying over P then it is isolated in its fiber. We shall return to this notion soon and explain the use of the word "isolated" here, but we first want to state Zariski's Main Theorem, which we sometimes abbreviate ZMT.

Theorem (Zariski's Main Theorem. *Suppose that $R \subseteq R[\theta_1, \ldots, \theta_n] \subseteq S$ are commutative rings and that R is integrally closed in S while S is integral over $R[\theta_1, \ldots, \theta_n]$. Let Q be a prime ideal of S that is isolated in its fiber over $P \in \mathrm{Spec}\,(R)$. Then there exists an element $f \in R - P$ such that the induced homomorphism $R_f \to S_f$ is an isomorphism.*

We want to examine some consequences of this result, including a very important geometric corollary, before we give the proof, which is difficult and lengthy.

We next review some basic algebraic geometry. Let K be an algebraically closed field. For simplicity, at this point we shall restrict our attention primarily to closed algebraic sets in some $\mathbb{A}_K^N$. Let X be such a set. Unless otherwise specified, when we refer to "points of X" we mean closed points. By a *variety* we mean a nonempty irreducible closed algebraic set in some $\mathbb{A}_K^N$. (As a scheme, a variety is reduced and irreducible.) We write $K[X]$ for the coordinate ring of the affine (closed) algebraic set X: it is the ring of regular functions from X to $K = \mathbb{A}^1_K$. It is a finitely generated reduced K-algebra (and, up to isomorphism, all such algebras occur). X is a variety iff $K[X]$ is a domain. Note that the category of (closed) affine algebraic sets over K and regular morphisms is anti-equivalent to the category of finitely generated reduced K-algebras and K-algebra homomorphisms: essentially, these are opposite categories. If X is a variety then the fraction field of $K[X]$ is denoted $K(X)$ and is called the *function field* of X. Its elements may be regarded as regular functions defined on some nonempty (equivalently, dense) open set in X, where two functions are equivalent if they agree on the intersection of their domains, which will be another dense open set. A morphism $g : X \to Y$ of varieties is called *dominant* if its image is dense. This holds if and only if the induced map of $K[Y] \to K[X]$ is injective, for that map has kernel containing I if and only if the image of g is contained $V(I) \subseteq Y$. A dominant map induces a map of function fields $K(Y) \to K(X)$, which is necessarily injective. By definition, the variety Y is normal precisely when $K[Y]$ is normal, i.e., integrally closed in its field of fractions $K(Y)$.

The following is a corollary of ZMT, and is also referred to as Zariski's Main Theorem. The restriction to affine varieties is not needed and is only made for simplicity. We shall explain how the Corollary is deduced in detail later.

Corollary (Zariski's Main Theorem). *Let $g : X \to Y$, be a morphism of affine varieties as in the preceding discussion. If g is bijective on closed points, Y is normal, and $K(Y)$ is separable over $K(X)$, then g is an isomorphism.*

We have not yet proved anything. We first want to discuss why some hypothesis other than having g be bijective on closed points is needed.

Let Y be $V(y^3 - z^2)$ in $\mathbb{A}^2_K$, which may also be described as the set $\{(\lambda^2, \lambda^3) : \lambda \in K\}$. Then $K[Y] \cong K[y, z]/(y^3 - z^2) \cong K[x^2, x^3] \subseteq K[x]$. Let $X = \mathbb{A}^1_K$. The map $X \to Y$ that sends λ to (λ^2, λ^3) is bijective. It corresponds to the map $K[Y] = K[y, z]/(y^3 - z^2) \cong K[x^2, x^3] \subseteq K[x] \cong K[X]$ that sends the images of y and z to x^2 and x^3, respectively. This is an example of a bijective map of varieties that is not an isomorphism: the problem, in some sense, is that $K[Y]$ is not normal — the element x is in its integral closure. Thus, Y is not normal. Note that the map of varieties cannot be an isomorphism because the corresponding map of K-algebras is not surjective, and therefore is not an isomorphism.

Even when both varieties are normal, or even regular, the separability condition is needed. Let K be an algebraically closed field of prime characteristic $p > 0$ and let $X = Y = \mathbb{A}^1_K$. Let $g : X \to Y$ be the map sending $\lambda \mapsto \lambda^p$. Since K is algebraically closed it is perfect, and so the map is surjective and therefore bijective. This morphism corresponds to the K-algebra map of rings $K[x] \to K[x]$ sending $x \mapsto x^p$, or to the inclusion $K[x^p] \subseteq K[x]$. The map of rings is not surjective and so g is not an isomorphism. The induced map of function fields is $K(x^p) \subseteq K(x)$, which is evidently not separable: that is the problem.

We next want to note that ZMT is rather non-trivial even in very special cases: it implies a key lemma that can be used to deduce Hilbert's Nullstellensatz very quickly. Suppose that in the statement of (the ring-theoretic form of) ZMT one assume that $R = K \subseteq L = S$

where K is an algebraic closed field and S is a field that is finitely generated as a K-algebra. Then ZMT applies: K is integrally closed in L because it is algebraically closed in L. Take $P = (0) \subseteq K$ and $Q = (0) \subseteq L$ as the two primes (of course, there are no other primes to choose). Evidently Q is isolated in its fiber, since L has only one prime. Then there exists $f \in K - P$ such that $K_f \cong L_f$. Since $f \neq 0$ it is already invertible in K and L, and so we see that $K = L$. That is, a field finitely generated as an algebra over an algebraically closed field K must be equal to K. This result is sometimes called Zariski's lemma. However, our proof of ZMT will not give a new proof of Hilbert's Nullstellensatz: we make use of Hilbert's Nullstellensatz in the argument.

We next want to explain the used of the word "isolated" in the expression "isolated in its fiber." A point x of a topological space X is called *isolated* if it is both open and closed in X. The fiber, as a topological space, is the Spec of the ring $A = \kappa_P \otimes_R S$, and so may be thought of as a topological space. We want to make two observations:

(1) If a prime m is an isolated point of $\mathrm{Spec}\,(A)$, then m is both maximal and minimal among the prime ideals of A.

(2) If A is Noetherian or, much more generally, if the prime ideal m is finitely generated, then m is an isolated point of $\mathrm{Spec}\,(A)$ if and only if m is both maximal and minimal in $\mathrm{Spec}\,(A)$.

To see this, first note that $\{m\}$ is closed if and only if m is maximal. Now $\{m\}$ is open if and only if there exists $f \in R$ such that mR_f is the unique prime ideal of R_f: this is the condition for $D(f) = \{m\}$. Since a prime has arbitrarily small open neighborhoods of the form $D(f)$, the set consisting of just that prime cannot be open unless it is equal to $D(f)$ for some choice of f. But if $m = D(f)$ it must be minimal: any strictly smaller prime would also be in $D(f)$. Finally, suppose that m is both maximal and minimal and that it is finitely generated, say by $u_1, \ldots, u_h$. If we localize at m it becomes the only prime of R_m, and so every u_j has become nilpotent. This implies that for every j we can choose $f_j \in R - m$ and N_j such that $f_j u^{N_j} = 0$. Let $f = f_1 \cdots f_h$. In the ring R_f, every generator of m is nilpotent. Since m is maximal, it is the only prime ideal of R_f, and thus $\{m\}$ is open as well as closed.

We next want to show how the geometric form of ZMT stated above follows from the algebraic form. We need the following basic facts about the behavior of dominant maps of varieties:

Lemma. *Let $g : X \to Y$ be a dominant map of algebraic varieties, so that we have an injection of domains $K[Y] \hookrightarrow K[X]$. Then:*

(a) *The transcendence degree of $K(X)$ over $K(Y)$ is $\delta = \dim(X) - \dim(Y)$.*

(b) *There is a dense open subset U of Y such that for every $u \in U$, the dimension of the fiber $g^{-1}(u)$, thought of as a closed algebraic set in X, is $\delta = \dim(X) - \dim(Y)$.*

(c) *If $\dim(Y) = \dim(X)$ then $K(X)$ is a finite algebraic extension of $K(Y)$. Assume also that $K(X)$ is separable over $K(Y)$. Then there is a dense open set $U \subseteq Y$ such that for all $u \in U$, the fiber $g^{-1}(u)$ is a finite set of cardinality $d = [K(X) : K(Y)]$.*

Proof. Given any three fields $K \subseteq \mathcal{F} \subseteq \mathcal{G}$ the transcendence degree of $\mathcal{G}$ over K is the sum of the transcendence degree of $\mathcal{F}$ over K and the transcendence degree of $\mathcal{G}$ over $\mathcal{F}$.

Part (a) follows from applying this to $K \subseteq K(Y) \subseteq K(X)$ along with the theorem that the dimension of a variety over K is the transcendence degree of its function field over K.

To prove part (b), let $R = K[Y] \subseteq K[X] = S$. Then S is a domain finitely generated over the domain R, and by the Noether normalization theorem for domains, we may localize at one nonzero element $f \in R$ so that S_f is a module-finite extension of a polynomial ring over R. The number of variables must be δ, the transcendence degree. Let U be the open set corresponding to $D(f)$ in Y. Thus, after replacing R and S by R_f and S_f, it suffices to show that if S is a module-finite domain extension of $R[x_1, \dots, x_\delta]$, then all fibers over maximal ideals m of R have dimension δ. Since S/mS is module-finite over $(R/m)[x_1, \dots, x_\delta]$ the dimension is at most δ. Since S has prime ideal Q lying over $mR[x_1, \dots, x_\delta]$ by the lying over theorem, and we have $S/mS \twoheadrightarrow S/Q$, while S/Q is a module-finite extension domain of $(R/m)]x_1, \dots, x_\delta]$, we also have that the dimension is at least δ.

It remains to consider part (c). We continue the notations from the proof of (b). The first statement is immediate from (a) and the fact that $S = K[X]$ is finitely generated over K and, hence, over $R = K[Y]$. We may localize at $f \in R - \{0\}$ and so assume that S is module-finite over R. Then $K(Y) \otimes_R S = K(X)$. Choose a primitive element θ for $K(X)$ over $K(Y)$: by multiplying by a suitable nonzero element in R, we may assume that θ is in S. Let G be the minimal monic polynomial of θ over the fraction field of R. By our hypothesis on the field extension, G will be separable over R. By inverting one more element of $R - \{0\}$ we may assume that the coefficients of G are in R. Note that $S/R[\theta]$, as an R-module, is torsion. Therefore we may invert yet another element of R and assume without loss of generality that $S = R[\theta]$, and then $S \cong R[x]/G$.

Consider the roots of G in a suitably large extension field of the fraction field of R. The product of the squares of their differences (the discriminant of G) is a symmetric polynomial over $\mathbb{Z}$ in the roots of G, and therefore is expressible as a polynomial D over $\mathbb{Z}$ in the coefficients of G, which, up to sign, are the elementary symmetric functions of the roots. The discriminant is therefore a nonzero element of R. We localize at the discriminant as well, and so we may assume that it is a unit of R. Note that each localization has the effect of restricting out attention to a smaller dense open subset of Y.

The points of the fiber over a m, a maximal ideal of R, correspond to the maximal ideals of $(R/m)[x]/\overline{G}$, where $\overline{G}$ is simply the image of G modulo m. But $R/m = K$ and the discriminant of $\overline{G}$ is simply the image of the discriminant of G (one substitutes the images of the coefficients of G into D), and so is not zero. It follows that the roots of G are mutually distinct, and so the number of points in the finite fiber is precisely the degree of G, which is the same as the degree of G and is equal to $[K(Y) : K(X)]$. $\square$