Math 711: Lecture of September 10, 2004

Proof of the geometric form of ZMT. We are now ready to deduce the geometric form of ZMT from the algebraic form. The fact that the map $g : X \to Y$ is bijective implies that it is dominant. Therefore, we may consider $R = K[Y] \subseteq K[X] = S$. We want to prove that $R = S$. Consider S/R as an R-module. If it is nonzero, then there is a maximal ideal m of R such that $(S/R)_m \neq 0$, and then $R_m \neq S_m$, where S_m is simply $(R - m)^{-1}S$. The bijectivity of the map shows that all set-theoretic fibers over closed points consist of exactly one closed point. From part (b) of the preceding Lemma, we must have $\dim(X) = \dim(Y)$, and then part (c) shows that the extension is algebraic of degree 1, which means that $K(X) = K(Y)$: note that we are using separability here. Since R_m is normal, it is integrally closed in S_m, for S_m is contained in the fraction field of R. Since S is finitely generated over R, S_m is finitely generated over R_m. The fiber S_m/mS_m contains just one prime ideal, which is isolated in its fiber. Therefore we can choose $f \in R_m - mR_m$ such that $(R_m)_f = (S_m)_f$. But since f is already invertible in R_m and S_m, this implies that $R_m = S_m$, a contradiction. $\square$

We are now ready to begin the proof of the algebraic form of ZMT. We have that $Q \in \operatorname{Spec}(S)$ is isolated in its fiber over $P \in \operatorname{Spec}(R)$, where $R[\theta_1, \ldots, \theta_n] \subseteq S$ is integral, while R is integrally closed in S. We shall used induction on n. If $n = 0$, then S is integral over R, and since R is integrally closed in S, this implies that $R = S$, and we are done.

We postpone considering the case where $n = 1$. Instead, we assume this case for the moment, and carry through the inductive step. This will reduce the problem to studying the case where $n = 1$. For this purpose let T denote the integral closure of $R[\theta_1, \ldots, \theta_{n-1}]$ in S. Let $\theta = \theta_n$. Note that S is integral over $T[\theta]$. Let $q = Q \cap T$. We want to show that q is isolated in its fiber over R, which is the fiber over P, since it is clear that $q \cap R = Q \cap R = P$. Suppose that we can do this. Then we can choose $f \in R - P$ such that $R_f = T_f$, by the induction hypothesis. Now Q is evidently isolated in its fiber over q, since q lies over P in R, and this is preserved when we localize at f. Also, S is integral over $T[\theta]$, and so S_f is integral over $T_f[\theta]$. Thus, we can apply the case $n = 1$ to conclude that there exists $\gamma = T_f - qT_f$ such that $(T_f)_\gamma = (S_f)_\gamma$. Now, $T_f = R_f$, and it follows that $q_f = PR_f$, so that we may write $\gamma = g/f^h$ where $g \in R - P$. This gives $R_{fg} = (R_f)_\gamma = (T_f)_\gamma = (S_f)_\gamma = S_{fg}$, as required.

It remains to prove that q is isolated in its fiber over P.

We need the following result (think of S_0 as $R[\theta_1, \ldots, \theta_n]$).

Lemma. *Let $R \subseteq T \subseteq S$ be rings. Let Q be a prime of S lying over $P = Q \cap R$, and let $q = Q \cap T$.*

(a) *If Q is minimal in its fiber over T and $u \in T - q$ is such that $T_u = S_u$, then q is also minimal in its fiber over P.*

(b) *If S is integral over S_0, $R \subseteq S_0 \subseteq S$ with S_0 finitely generated over R, and Q is maximal in its fiber over T, then q is maximal in its fiber over P.*

(c) *If Q is isolated in its fiber, S is integral over a finitely generated R-subalgebra S_0, and there exists $u \in T - q$ such that $T_u = S_u$, then q is isolated in its fiber over T.*

Proof. To prove part (a) note that if q' is strictly contained in q and q' lies over P, we may expand q' to $T_u = S_u$ and then contract to S to obtain a prime Q' of S strictly smaller than Q and lying over P: we have a bijection between primes of T not containing u and prime of S not containing u.

For part (b), first replace R, T, and S by their localizations at P. Thus, we may assume that R is local with maximal ideal P. Then $R/P \subseteq T/q \subseteq S/Q$, and we also have $R/P \subseteq S_0/q' \subseteq S/Q$, where $q' = Q \cap S_0$. The fact that S is maximal in its fiber implies that S/Q is a field. Since S/Q is integral over S_0/q', S_0/q' is also a field, and it is finitely generated over R/P. It follows that S_0/q' is a finite algebraic extension of R/P. Now we get that S/Q is an algebraic extension of R/P. It now follows that T/q is a field, and this means that q is maximal in its fiber.

Part (c) is simply the result of combining (a) and (b). $\square$

We can now use this Lemma to to see that q as defined earlier is isolated in its fiber over P. We have the hypothesis of part (b). The hypothesis of part (a) also holds, because we may apply the case $n = 1$ to the inclusion $T \subseteq S$ (Q is obviously isolated in its fiber over q) to obtain $u \in T - q$ such that $T_f = S_f$). The result we need is now immediate from part (c). This completes the reduction to the case where $n = 1$. $\square$