Math 711: Lecture of September 17, 2004

For the remainder of the proof we shall be studying the case where $R \subseteq R[\theta] \subseteq S$. R is integrally closed in S, S is integral over $R[\theta]$, and Q, a prime of S, is isolated in its fiber over $P = Q \cap R$. We want to reduce to the case where S is actually module-finite over $R[\theta]$. For this purpose it will suffice to find T module-finite over $R[\theta]$ such that $R[\theta] \subseteq T \subseteq S$ and $q = Q \cap T$ is isolated in its fiber over P. For once we have T as specified, if we know the theorem in the module-finite case we can choose $f \in R - P$ such that $R_f = T_f \subseteq S_f$. Then R_f is integrally closed in S_f and S_f is integral over $T_f = R_f$ together imply that $R_f = S_f$. Quite generally, we are free to replace S by any ring T with $R[\theta] \subseteq T \subseteq S$ such that $Q \cap T$ is minimal in its fiber: by the argument just given, if the desired result holds for T then it holds for S.

Note that by part (b) of the Lemma of September 10, we have that q is maximal in its fiber for any choice of T with $R[\theta] \subseteq T \subseteq S$. Thus, the problem is to choose T such that q is minimal in its fiber. Thus, we want to find $s_0, s_1, \ldots, s_h \in S$ such that with $T = R[\theta][s_0, s_1, \ldots, s_h]$, $q = Q \cap T$ is minimal in its fiber over P. For this purpose we may replace R, S by R_P, S_P and θ by its image in S_P. If we find elements $s_0/f_0, s_1/f_1, \ldots, s_h/f_h$ that work here, where the $s_j \in S$ and the $f_j \in R - P$, we may use $s_0, s_1, \ldots, s_h$, since the f_j become invertible in any case when we calculate the fiber. Let $K = R/P$. Let $\overline{\theta}$ be the image of θ in $R[\theta]/PR[\theta] \cong K[\overline{\theta}]$. If $\overline{\theta}$ satisfies a monic polynomial of positive degree over K, then we may take $T = R[\theta]$: the fiber is module-finite over K and, hence, zero-dimensional, and so all primes of the fiber of T over P will be isolated in that case. Therefore, we may assume without loss of generality that $\overline{\theta}$ is transcendental over K. Q lies over a prime of this ring that is maximal in its fiber, by part (b) of the Lemma of September 10, and this prime will be generated by $g(\overline{\theta})$, where g is a monic polynomial over K. Lift this element to an element $\gamma \in R[\theta]$. Then $\gamma \in Q$ is nilpotent mod PS, since Q is minimal in its fiber.

Hence, we can choose $s_0 \in S - Q$, $p_1, \ldots, p_h \in P$, and $s_1, \ldots, s_h \in S$ such that $s_0\gamma^N = \sum_{j=1}^{h} s_j p_j$. Choose $T = R[\theta][s_0, s_1, \ldots, s_h]$. We claim that $q = Q \cap T$ is minimal in its fiber over P, for a smaller prime q_0 that lies over P will not contain s_0, and so must contain γ. But then q and q_0 contract to the same prime in $R[\theta]$, since there is only one prime that contains P and γ, and since T is integral over $R[\theta]$ it follows that $q = q_0$.

Once we have replaced S by a module-finite extension of $R[\theta]$, we may again replace R by R_P and S by S_P, and it suffices to see that $R_P = S_P$. For if these two are equal, then for each of the finitely many generators of S over R we may choose f_j multiplying that generator into R with $f_j \in R - P$. Let f be the product of the f_j. Then $R_f = S_f$. Henceforth we assume that $R = (R, P, K)$ is quasi-local as well.

If $A \subseteq B$ is module-finite the *conductor* of B in A is defined as $\{a \in A : Ba \subseteq A\}$. It is readily checked to be an ideal of A and an ideal of B. It is also easy to see that it is the largest ideal of A that is also an ideal of B, since any element a of such an ideal has the property that $Ba \subseteq A$.

Throughout the rest of the proof of ZMT, let J be the conductor of S in $R[\theta]$. The remainder of the proof breaks up into two cases: one where $J \not\subseteq Q$, which is easier, and one where $J \subseteq Q$. The second case will require two preliminary results.

We first discuss the case where $J \not\subseteq Q$. Let $u \in J - Q$. Then $R[\theta]_u = S_u$, and by part (a) of the Lemma of September 10, $Q \cap R[\theta]$ is isolated in its fiber over P. We may therefore replace S by $R[\theta]$: once we have the result for $R[\theta]$, the case of S follows, by the comment at the end of the first paragraph of the preceding page.

But then $R[\theta]/PR[\theta] = K[\overline{\theta}]$ cannot be a polynomial ring in $\overline{\theta}$ over K, for no prime could then be isolated. It follows that there is a polynomial in θ with at least one coefficient not in P whose value is in $PR[\theta]$. Subtracting, we find that there is a polynomial in θ over R that vanishes and has at least one coefficient not in P. Choose such a polynomial of least degree d, and call it $r_d\theta^n + r_{d-1}\theta^{d-1} + \cdots + r_0$. By multiplying the polynomial by r_d^{d-1}, we see that $r_d\theta$ is integral over R and therefore in R. Thus, we may re-write the polynomial as $(r_d\theta + r_{d-1})\theta^{d-1} + \cdots + r_0$, which has lower degree in θ. Therefore all of its coefficients are in P, and we conclude that $r_d\theta + r_{d-1} \in P$. If r_d is a unit, we find that $\theta \in R$, as required. If $r_d \in P$ and r_{d-1} is a unit, we find that $r_{d-1} \in PS$, a contradiction, since Q contains PS. This completes the proof for the case where $J \not\subseteq Q$.

In the remaining and most difficult case, where $J \subseteq Q$, we shall show that Q cannot be isolated in its fiber. We need two preliminary results.

Lemma. *If $R \subseteq R[\theta] \subseteq S$ are domains with S integral over $R[\theta]$ and θ is transcendental over R, then no prime ideal of S is isolated in its fiber.*

Proof. Suppose that Q is isolated in its fiber over $P = S \cap R$. Let S' be the integral closure of S in its fraction field. By the lying over theorem, there exists a prime ideal Q' of S' lying over Q. Q' must also be isolated in its fiber over P: a prime Q'_1 comparable to but distinct from it lying over P would yield such a prime lying over P and comparable to but distinct from Q when contracted to S (Q'_1 cannot lie over Q: primes lying over the same prime in an integral extension are mutually incomparable). Henceforth we assume that $S = S'$ is integrally closed. It then contains an integral closure R' of R in the fraction field of R. Then Q will be isolated in its fiber over $P' = Q \cap R'$: a comparable prime lying over P' will automatically lie over P (note that $P' \cap R \subseteq Q \cap R = P$). Therefore we may replace R by R' and $R[\theta]$ by $R'[\theta]$, and so assume that R is integrally closed. But now both going up and going down hold between $R[\theta]$ and S, since $R[\theta]$ is again normal, and it follows that $Q \cap R[\theta]$ is also isolated in its fiber over P. We may consequently replace S by $R[\theta]$. But now we can see that no prime is isolated in its fiber, since the fiber over P is the polynomial ring in one variable over a field, and there is no prime that is both maximal and minimal. $\square$