Math 711: Lecture of September 22, 2004

It is obvious from the definition that a homomorphism is étale if and only if it is both smooth and unramified, and that is also obvious from our characterizations using differentials.

Note also that the composition of two smooth (respectively, étale, respectively, unramified) homomorphisms is again smooth (respectively, étale, respectively, unramified). For example, suppose that S_2 is smooth over S_1 and S_1 is smooth over R. Given a map $S_2 \to T/J$ where $J^2 = 0$ we have a composite map $S_1 \to S_2 \to T/J$ which lifts to a map $S_1 \to T$. Now, since T is an S_1-algebra and S_2 is smooth over S_1 we can lift to a map $S_2 \to T$. Likewise, given two R-algebra liftings of an R-algebra map $S_2 \to T/J$ to two maps $S_2 \to T$ that are distinct, the induced maps $S_1 \to S_2 \to T$ must also be distinct, since S_2 is unramified over S_1. But these both lift the same R-algebra map $S_1 \to T/J$, contradicting the fact that S_1 is unramified over R. The corresponding result for étale maps is the consequence of putting these two results together.

We want to give a bit more feeling for derivations and modules of differentials. Note that by a straightforward induction, if D is a derivation on R then

$$D(f_1 f_2 \cdots f_k) = f_2 \cdots f_k D(f_1) + \cdots + f_1 f_2 \cdots f_{k-1} D(f_k).$$

A typical term in the sum is the product of all the f_i except f_j times $D(f_j)$. An immediate consequence is that $D(f^k) = kf^{k-1}D(f)$. It then follows that $D(f_1^{k_1} \cdots f_h^{k_h})$ is the sum of the h terms of which a typical term is $k_j f_1^{k_1} \cdots f_j^{k_j - 1} \cdots f_h^{k_h} D(f_j)$. It also follows that the value of an A-derivation D on a polynomial in $f_1, \ldots, f_h$ with coefficients in A is uniquely determined by the values of $D(f_1), \ldots, D(f_h)$.

In the case of the a polynomial ring $B[x]$ there is B-derivation $\dfrac{\partial}{\partial x}$ (we use the partial derivative notation because B may be a polynomial ring involving other variables) whose value on $F(x)$ is

$$\left.\frac{F(x+\Delta) - F(x)}{\Delta}\right|_{\Delta=0}$$

where Δ is a new indeterminate. For $b \in B$, $\dfrac{\partial}{\partial x}(bx^k) = kbx^{k-1}$.

The polynomial ring R over A in variables x_i (there may be infinitely many) has a derivation $\dfrac{\partial}{\partial x_i}$ for each variable x_i: one may think of R as $B_i[x_i]$ where $B_i = A[x_j : j \neq i]$. Then if R is an A-algebra, $F \in A[X_1, \ldots, X_k]$ and $f_1, \ldots, f_k \in R$, for any A-derivation D we have that

$$D\big(F(f_1, \ldots, f_k)\big) = \sum_{i=1}^{k} \frac{\partial F}{\partial x_i}(f_1, \ldots, f_k) D(f_i)$$

.

It follows that if we have elements f_j that generate R over A then the elements df_j span $\Omega_{R/A}$ as an R-module. In particular, if R is a finitely generated A-algebra then $\Omega_{R/A}$ is a finitely generated R-module.

Given a polynomial ring R in variables x_i over A, a derivation D of R into M is uniquely determined by specifying values $u_i \in M$ for the variables x_i, and it is straightforward to check that there really is a derivation for each specified set of values $\{u_i\}_i$: it sends F to

$$\sum_i \frac{\partial F}{\partial x_i} u_i.$$

This implies that for the polynomial ring R, $\Omega_{R/A}$ is the free R-module on the dx_i. This is true whether the number of variables is finite or infinite.

Note that if we have an A-algebra homomorphism $R \to S$ and an S-module M there is an R-linear map $\mathrm{Der}_A(S,\, M) \to \mathrm{Der}_A(R,\, M)$ (where M is thought of as an R-module via restriction of scalars) that is simply induced by composition with the map $R \to S$. This implies that there is an R-linear map $\Omega_{R/A} \to \Omega_{S/A}$ sends df (this might more precisely be denoted $d_{R/A}f$) to df (which might more precisely be denoted $d_{S/A}f$). Hence, there is an S-linear map $S \otimes_R \Omega_{R/A} \to \Omega_{S/A}$.

Let I be an ideal of the A-algebra R. Given an A-derivation $R/I \to M$ we may compose to get an A-derivation $R \twoheadrightarrow R/I \to M$. An A-derivation $D : R \to M$ arises in this way if and only if M is an (R/I)-module and D kills I. It follows that if we have an A-algebra surjection $R \twoheadrightarrow S$ with kernel I, then $\Omega_{S/A} = S \otimes_R \Omega_{R/A}/\mathrm{Span}\,\{df : f \in I\}$. In the denominator here, it suffices to kill the R-span of the df_j for a set of elements f_j that generate I (note that $d(rf_j) = rd(f_j) + f_j dr$, and the second summand is 0 because I kills M).

Any A-algebra R may be thought of as a polynomial ring T in variables x_i modulo the ideal generated by polynomials F_j: both i and j may vary in an infinite set here. It follows that $\Omega_{R/A}$ may be thought of as the free R-module on the dx_i modulo the images of the elements dF_j calculated in $\Omega_{T/A}$, i.e., the images of the elements $\sum_i \frac{\partial F_j}{\partial x_i} dx_i$.

In the case of a finitely presented A-algebra R we may work with finitely many variables x_i and finitely many polynomials F_j, and then $\Omega_{R/A}$ is the cokernel of the matrix $\left(\dfrac{\partial F_j}{\partial x_i}\right)$ (each entry is replaced by its image in R). This matrix is called the *Jacobian matrix.* The Jacobian matrix depends on a choice of generators and relations for R over A, but its cokernel, $\Omega_{R/A}$, does not.

If R is an A-algebra, W is a multiplicative system in R, and $D : R \to M$ is a derivation, then any derivation $D : R \to M$ induces a unique derivation $W^{-1}D : W^{-1}R \to W^{-1}M$ whose restriction to R is D. Since $w(f/w) = f/1$ for $f \in R$ and $w \in W$, the extended derivation, if there is one, $\widetilde{D}$, must satisfy

$$D(f)/1 = \widetilde{D}(f) = \widetilde{D}\big(w(f/w)\big) - w\widetilde{D}(f/w) + (f/w)\widetilde{D}(w/1) = w\widetilde{D}(f/w) + (f/w)\big(D(w)/1\big).$$

We may multiply by $1/w$ to obtain

$$\widetilde{D}(f/w) = \frac{D(f)}{w} - \frac{fD(w)}{w^2} = \frac{wD(f) - fD(w)}{w^2},$$

the usual quotient rule. This proves that there is at most one way to define $\widetilde{D}$. It is straightforward to check that if one takes this as the definition (one must check that if $f_1/w_1 = f_2/w_2$ then one gets the same result from either of these representations) then one does in fact get a derivation that extends D. The remaining details are also straightforward.

It then follows easily from the universal mapping properties for the modules of differentials and for localization that

$$\Omega_{W^{-1}R/A} \cong W^{-1}\Omega_{R/A}.$$

Before proving the theorem we have already stated characterizing smooth, étale, and unramified morphisms in terms of the behavior of differentials, we want to give some further characterizations: the proofs of these results are likewise postponed for a while.

We need the notion of a geometrically regular algebra over a field. If K is a field, we say that a Noetherian K-algebra R is *geometrically regular* if for every finite purely inseparable extension L of K, the ring $L \otimes_K R$ is regular. This implies that R is regular, and in equal characteristic 0, it is equivalent to the condition that R be regular, since the only purely inseparable extension of K is K. Similarly, if K is algebraically closed or perfect, the condition that a K-algebra be geometrically regular is, again, simply the condition that it be regular. It turns out that if R is geometrically regular over K, then $L \otimes_K R$ is regular for every finite algebraic extension of K. In this generality, infinite field extensions L are slightly problematic in that one may lose the Noetherian property.

However, when R is essentially of finite type over K, i.e., a localization of a finitely generated K-algebra, the geometric regularity of R over K implies that $L \otimes_K R$ is regular for every field extension L of K. In this case it turns out that R is geometrically regular provided that $L \otimes_R K$ is regular for some field that contains a perfect closure of K (in characteristic p, this is a maximal purely inseparable algebraic extension, gotten by adjoining all p^e th roots of all elements). In particular, if R is essentially of finite type over K, then R is geometrically regular if and only if $\overline{K} \otimes_K R$ is regular, where $\overline{K}$ is an algebraic closure of K.

Note the following example. Let $R = k(t)$ be a field where k is a field characteristic $p > 0$ and t is transcendental over k. Let $K = K(t^p) \subseteq R$. K is a field, and R is a regular, but it is not a geometrically regular K-algebra: $k(t) \otimes_K R$ contains the nilpotent $t \otimes 1 - 1 \otimes t$, whose pth power is 0.

We will eventually prove all of the statements about geometric regularity made above. But we first want some additional characterizations of smooth, étale, and unramified morphisms, of which the next is:

Theorem. *Let S be a finitely presented R-algebra.*
(a) *S is smooth over R if and only if S is flat over R and for every prime P of R, the fiber $\kappa_P \otimes_R S$ is geometrically regular over κ_P.*
(b) *S is étale over R if and only if S is flat over R and for every prime P of R, the fiber $\kappa_P \otimes_R S$ is a finite product of finite separable algebraic field extensions of κ_P.*

Note that a smooth algebra over a field K is the same as a finitely generated geometrically regular algebra, while an étale algebra over a field K is the same thing as a finite product of finite separable algebraic field extensions.

For varieties over the complex numbers $\mathbb{C}$, étale has the following geometric interpretation: $R \to S$ is étale means that the map of corresponding varieties is, locally, like an open inclusion in a covering space with finite fibers. The interpretation of unramified is that the geometric map is like a locally closed inclusion in a covering space with finite fibers.

In the algebraic setting we now give very down-to-earth characterizations of all these notions. Let R be a ring and let x be an indeterminate over R. Let f be a monic polynomial in x, and let $g \in R[x]$ be such that f', the derivative of f with respect to x, is invertible in $R[x]_g$. It then turns out that $(R[x]/fR[x])_g$ is étale over R. Such an extension is called a *standard étale* R-algebra. The following is a deep result characterizing étale and unramified extensions:

Theorem. *Let S be a finitely presented R-algebra. Then S is étale (respectively, unramified) over R if and only if for every prime ideal Q of S with contraction P to R there exist $b \in S - Q$ and $a \in R - P$ such that S_b is isomorphic to a standard étale algebra over R_a (respectively, a homomorphic image of a standard étale algebra over R_a).*

The result just stated requires ZMT in the proof, and generalizes immensely that statement that a finite separable algebraic field extension has a primitive element.

There is a similar result characterizing smooth homomorphisms:

Theorem. *Let S be a finitely presented R-algebra. Then S is smooth over R if and only if for every prime ideal Q of S with contraction P to R there exist $b \in S - Q$ and $a \in R - P$ such that there is a factorization $R_a \to T \to S_b$, where T is a polynomial ring in finitely many variables over R_a and $T \to S_b$ is étale.*

If we are working with varieties over $\mathbb{C}$ then $R \to S$ is smooth means that, locally on $\mathrm{Spec}\,(R)$ and $\mathrm{Spec}\,(S)$, the map factors as an open inclusion in a covering space with finite fibers of the product of the base with $\mathbb{A}^m_{\mathbb{C}}$ for some m.

Filling in the proofs of the results we have stated will be a considerable task.