Math 711: Lecture of September 24, 2004

An R-algebra S is called *formally smooth*, respectively *formally étale*, respectively *formally unramified* if for all R-algebras T and ideals $J \subseteq T$ such that $J^2 = 0$, the map $\Theta_{T,J}$ is surjective, respectively bijective, respectively injective. Evidently, in each case, the word "formally" may be dropped, if the property of finite presentation for S over R is assumed as well. If two homomorphisms $f : R \to S$ and $g : S \to T$ are formally smooth (or formally étale, or formally unramified), then their composition $g \circ f : R \to T$ has the same property.

Proposition. *Let S be an R-algebra. If S is a polynomial ring over R, then S is formally smooth, and it is smooth if the number of indeterminates is finite. If $S = W^{-1}R$ then S is formally étale over R, and it is étale if W is finitely generated. If $S = R/I$ then S is formally unramified over R, and it is unramified if I is finitely generated.*

Proof. For the result on polynomial rings, note that the values on the indeterminates x_i are elements $\overline{t}_i$ of T/J, where $\overline{t}_i$ is the image mod J of $t_i \in T$. One may lift the map by sending $x_i \mapsto t_i$ for all i. This does not use that $J^2 = 0$.

Given a map of $f : W^{-1}R \to T/J$, one has a map of $R \to T/J$, and this has a unique lifting to a map $R \to T$. The map $W^{-1}R \to T$ that lifts f must extend this map, and there is at most one map that does so: it exists if and only if every element of W maps to a unit of T. But this is true, because every element of W maps to a unit of T/J, and killing nilpotents does not affect the invertibility of elements.

Finally, if an R-algebra map $R/I \to T/J$ has two liftings to maps $R/I \to T$, these will induce, by composition with $R \to R/I$, distinct R-algebra maps $R \to T$, a contradiction. $\square$

We next note that if S is formally smooth, or étale, or unramified over R, and if R' is any R-algebra, then $R' \otimes_R S$ is formally smooth, or étale, or unramified over R'. The same result holds with the word "formally" omitted, since if S is finitely presented over R then $R' \otimes_R S$ is finitely presented over R': if $S = R[x_1, \ldots, x_n]/(F_1, \ldots, F_m)$ then $R' \otimes_R S \cong R'[x_1, \ldots, x_n]/(F'_1, \ldots, F'_m)$, where F'_j is the image of F_j in $R'[x_1, \ldots, x_n]$. The point in the proofs is that if T is an R'-algebra, then $\mathrm{Hom}_{R'-\mathrm{alg}}(R' \otimes_R S, T) \cong \mathrm{Hom}_{R-\mathrm{alg}}(S, T)$ as sets, and the same holds when T is replaced by T/J. The required results are then immediate.

We shall say that $\delta : R \to M$, where M is an R-module, is a *universal derivation* for the A-algebra R if for every derivation $D : R \to N$, where N is an R-module, there is a unique R-linear map $L : M \to N$ such that $D = L \circ \delta$. We have already noted that $d : R \to \Omega_{R/A}$ is a universal derivation. If $\delta : R \to M$ is another, the universal mapping properties give unique R-linear maps $L : \Omega_{R/A} \to M$ and $L' : M \to \Omega_{R/A}$ that are mutually inverse (e.g., $L' \circ L$ is the unique map from $\Omega_{R/A}$ to itself whose composition with d gives d, and so is the identity). Note that $\delta = L \circ d$.

If S is an R-algebra, let I be the kernel of the map $S \otimes_R S \to S$. The ideal I is generated by elements of the form $s \otimes 1 - 1 \otimes s$. Now $S \otimes_R S$ has two S-module structures coming from the two maps of S into it (one structural morphism maps s to $s \otimes 1$ and the other maps s to $1 \otimes s$), and every ideal of $S \otimes_R S$ has these two S-module structures. In particular, I and I^2 have two S-module structures. However, we claim that on I/I^2 these two S-module structures agree. The reason is that

$$I/I^2 = \big(S \otimes_R S)/I\big) \otimes_{S \otimes_R S} I$$

is an $\big((S \otimes_R S)/I\big)$-module, and $(S \otimes_R S)/I = S$. Notice that we have a map $\delta : S \to I$ such that for all $s \in S$, $\delta(s) = s \otimes 1 - 1 \otimes s$. This map is easily seen to be R-linear. In fact, it is an R-derivation of $S \to I/I^2$, since

$$s\delta(t) + t\delta(s) = s(t \otimes 1 - 1 \otimes t) + t(s \otimes 1 - 1 \otimes s).$$

In evaluating $s(f \otimes g)$ we may use either $sf \otimes g$ or $f \otimes sg$. Thus, this expression becomes

$$st \otimes 1 - s \otimes t + s \otimes t - 1 \otimes st = st \otimes 1 - 1 \otimes st = \delta(st),$$

as required.

Theorem. *With notation as just above, $\delta : S \to I/I^2$ is a universal R-derivation on S, and so $\Omega_{S/R} \cong I/I^2$ in such a way that ds corresponds to $s \otimes 1 - 1 \otimes s$.*

Proof. Let $S = K[X_i : i]/(F_j : j)$ be a presentation of S, where i and j are both permitted to vary in index sets that may be infinite. We shall think of this as the copy of S on the right in $S \otimes_R S$. Let Y_i be a new family of indeterminates indexed in the same way as the X_i and let $G_j = F_j(Y)$ be the corresponding family of polynomials in the Y_i, so that $S \cong R[Y : i]/(G_j : j)$ as well, which we shall think of as the copy of S on the left in the $S \otimes_R S$. Then

$$S \otimes_R S \cong R[X_i,\, Y_i : i]/(F_j, G_j : j).$$

Let $\Delta_i = Y_i - X_i$ for every i. Then we may describe $S \otimes_R S$ using the indeterminates X_i and Δ_i, replacing Y_i by $X_i + \Delta_i$ for al i. We replace $G_j = F_j(Y)$ by $F_j(X + \Delta)$: we use this notation to indicate that in F_j, every X_i has been replaced by $X_i + \Delta_i$. In this presentation, if x_i is the image of X_i in S, then $\delta(x_i)$ is the image of $Y_i - X_i = \Delta_i$. Thus, the ideal I corresponds to the ideal generated by all the Δ_i in

$$R[X_i,\, \Delta_i : i]/(F_j, F_j(X + \Delta) : j).$$

By the multi-variable version of Taylor's formula, for all j,

$$F_j(X + \Delta) = F_j(X) + \sum_i \frac{\partial F_j}{\partial X_i}\Delta_i + \text{ terms of degree 2 or more in the } \Delta_i\ .$$

This leads to the result that

$$(S \otimes_R S)/I^2 \cong R[X_i, \Delta_i : i]/\big((F_j : j) + (\sum_i \frac{\partial F_j}{\partial X_i}\Delta_i : j) + (\Delta_i : i)^2\big).$$

Since $R[X_i : i]/(F_j : j) \cong S$, $(S \otimes_R S)/I^2$ may be identified with the free S module with basis 1 together with the Δ_i modulo the S-span of the relations $\sum_i \frac{\partial F_j}{\partial X_i} \Delta_i$ as j varies, where each $\frac{\partial F_j}{\partial X_i}$ is identified with its image in S. It follows that I/I^2 may be identified with quotient of the free S-module on the elements Δ_i modulo the S-span of those same relations, and we have already seen (see the fifth paragraph of the second page of the Lecture Notes of September 22) that this module is isomorphic with $\Omega_{S/R}$ in such a way that Δ_i corresponds to dx_i. $\square$

Theorem. *Let S be an R-algebra. Then S is formally unramified over R if and only if $\Omega_{S/R} = 0$, and this is equivalent to the condition that, with $I = \mathrm{Ker}\,(S \otimes_R S \twoheadrightarrow S)$, $I = I^2$.*

Proof. Since $\Omega_{S/R} \cong I/I^2$, it is obvious that $\Omega_{S/R} = 0$ if and only if $I = I^2$. We work with the latter condition.

We first show that if $I = I^2$ then S is unramified over R. Suppose that $I = I^2$ and also suppose we have two R-algebra maps ϕ_1 and ϕ_2 of $S \to T$ that agree mod J, with $J^2 = 0$. This gives an R-algebra map $S \otimes_R S \to T$ such that $s \otimes t \mapsto \phi_1(s)\phi_2(t)$. The fact that the ϕ_i induce the same map to T/J implies that for all $s \in S$, $s \otimes 1 - 1 \otimes s$ maps to 0 in T/J, and this implies that I maps into J. But then $I = I^2$ maps into $J^2 = 0$, and so I is killed. Since $s \otimes 1 - 1 \otimes s \in I$ maps to $\phi_1(s) - \phi_2(s)$, it follows that $\phi_1 = \phi_2$.

Conversely, suppose that S is formally unramified over R. Then let $T = (S \otimes_R S)/I^2$, and let $J = I/I^2 \subseteq T$. We have two obvious R-algebra maps of S into $S \otimes_R S$ (one sending s to $s \otimes 1$, and one sending s to $1 \otimes s$, and, hence, two obvious maps into T. Since the two maps agree mod J, they agree. But this means that each element $s \otimes 1 - 1 \otimes s$ is 0 in T/I^2, and since these elements generate I, we have that $I \subseteq I^2$. since the other inclusion is obvious, $I = I^2$.