Math 711: Lecture of September 27, 2004

Proposition. *Let R be a ring.*
(a) *If $S = R[x]_g/(f)$ where f is any polynomial whose derivative f' with respect to x is invertible in S, then S is étale over R.*
(b) *More generally, if $S = R[x_1, \ldots, x_n]_g/(f_1, \ldots, f_n)$ is such that the image of the Jacobian determinant $\det\big(\frac{\partial f_j}{\partial x_i}\big)$ is invertible in S, then S is étale over R.*

Proof. It is evident that (a) is a special case of (b) and it will suffice to prove (b). Suppose that we are given a homomorphism $\phi : S \to T/J$ where $J^2 = 0$ and we seek a lifting to T. Let $y_i \in T$ lift the values of the $\phi(x_j)$, where x_j is the image of X_j in S. Then we must have that $f_j(y) \in J$ for every j, where $y = y_1, \ldots, y_n$, and we also know that $g(y)$ is a unit of T, since it is a unit of T/J. Then we want to prove that there are unique elements $\delta_i \in J$ such that $f_j(y+\delta) = 0$ for all j, where $y + \delta$ indicates $y_1 + \delta_1, \ldots, y_n + \delta_n$. By Taylor's formula with remainder, the fact that the δ_j will be chosen in J, and $J^2 = 0$, these equations are equivalent to the equations

$$f_j(y) + \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}\bigg|_{x=y} \delta_i = 0,$$

with the proviso that the $\delta_i \in J$. Let $\mathcal{J}$ be the $n \times n$ matrix whose i, j entry is $\frac{\partial f_j}{\partial x_i}\Big|_{x=y}$. The determinant of this matrix is a unit, because that is true mod J, and so the matrix is invertible. Let Δ be a column vector whose entries are the unknown elements δ_i of J that we seek, and let Γ be a column vector whose j th entry is $f_j(y) \in J$. Thus, we seek to solve $\mathcal{J}\Delta = -\Gamma$ where Δ has unknown entries in J, $-\Gamma$ has entries in J, and the matrix $\mathcal{J}$ is invertible. It is now clear that the unique solution is $\Delta = -\mathcal{J}^{-1}\Gamma$, and the entries of the solution do happen to be in J. $\square$

Note that part (a) has the desirable consequence that standard étale extensions are, indeed, étale.

Corollary. *A finite separable algebraic extension L of a field K is étale over K.*

Proof. By the theorem on the primitive element, $L = K[\theta]$, where the monic minimal polynomial f of θ is separable over L. This implies that the image of f' in L does not vanish, and so is invertible. Thus $L \cong K[x]/(f)$ where f' is invertible in L. $\square$

Proposition. *If J is contained in the ideal of nilpotents of T, there is a bijection between the idempotents of T and the idempotents of T/J induced by the quotient surjection $T \twoheadrightarrow T/J$.*

Proof. Clearly, the image of an idempotent $e \in T$ is idempotent in T/J. Suppose that $e' \equiv e \bmod J$. Then $e - e'$ is nilpotent, and hence so is $e(e-e')$, i.e., $e = e^2 \equiv ee' \bmod J$, and so $e(1-e')$ is nilpotent. But e and $1-e'$ are both idempotent, and, hence, so is their product. It follows that $e(1-e')=0$, i.e., that $e=ee'$. But $ee'=e'$ similarly. Thus, the map on idempotents is injective.

It remains to show that it is surjective. Let e be an element whose image mod J is idempotent and let $f=1-e$. Then $e+f=1$ and $e^nf^n=0$ for some n. Expand $(e+f)^{2n-1}$ by the binomial theorem. Each term is either a multiple of e^n or a multiple of f^n. The multiples of e^n include e^{2n-1} and other terms involving f, and similarly for the multiples of f^n. Thus,

$$1=(e+f)^{2n-1}=e^n(e^{n-1}+fu)+f^n(f^{n-1}+ev).$$

Let $e'=e^n(e^{n-1}+fu)$ and $f'=f^n(f^{n-1}+ev)$. Then $e'+f'=1$, $e'f'$ is a multiple of $e^nf^n=0$, and, mod J, $e'\equiv e^{2n-1}+e^nfu\equiv e$. Thus, e' is an idempotent of T that lifts e. $\square$

Proposition. *Let $S_1, \ldots, S_n$ be R-algebras. Consider any of the following properties: finite presentation, (formal) smoothness, being (formally) étale, or being (formally) unramified. Then $S=S_1\times\cdots\times S_n$ has this property if and only if all of the S_i have this property.*

Proof. By a straightforward induction it suffices to consider the case where $n=2$. We leave the property of finite presentation as an exercise. Once that is known, it suffices to consider the properties of being formally smooth and formally unramified: the property of being formally étale then follows. Note that giving a map $S_1\times S_2\to T/J$ yields an idempotent in T/J that lifts uniquely to an idempotent in T, and so $T=T_1\times T_2$. We may then write $J=J_1\times J_2$ where J_i is an ideal of T_i and $J_i^2=0$, $i=1, 2$. The problem of lifting a map is a componentwise problem, and so if both factors are smooth (respectively, unramified) so is the product. Now suppose that S is smooth (respectively, unramified). Let T_1 be an S_1-algebra and J_1 an ideal such that $J_1^2=0$. Let $T=T_1\times S_2$ with $J_2=0$, and let $J=J_1\times J_2$. The problem of lifting maps $S_1\to T_1/J$ to maps $S_1\to T$ is equivalent to the problem of lifting maps $S_1\times S_2\to T/J$ of the form $\phi\times\mathbf{1}_{S_2}$ to maps $S_1\times S_2\to T$. It follows that if S is formally smooth (respectively, unramified), then so is S_1, and the argument for S_2 is similar. $\square$

We are aiming next to prove the following characterization of étale extensions of fields.

Theorem. *Let K be a field and let R be a finitely generated K-algebra. The following conditions are equivalent:*
(a) *R is étale over K.*
(b) *R is unramified over K.*
(c) *R is a finite product of finite separable algebraic field extensions of K.*
(d) *If L is an algebraic closure of K, then $L\otimes_K R$ is K-isomorphic with a finite product of copies of L.*

We postpone the proof until we have established some preliminary results on modules of differentials for field extensions. These results themselves need further preliminaries.

Let $K\subseteq L$ be a field extension. A family of algebraically independent elements $\{x_i\}_i$ of L over K is called a *separating transcendence basis* for L over K if L is separable over $K(x_i : i)\subseteq L$. We need the following result of S. MacLane:

Theorem. *If K is algebraically closed or perfect of characteristic $p > 0$ and L is finitely generated over K then L has a separating transcendence basis over K.*

Proof. If F is a subfield of L, let F^{sep} denote the separable closure of F in L. Choose a transcendence basis $x_1, \ldots, x_n$ so as to minimize $[L : L']$ where $L' = K(x_1, \ldots, x_n)^{\mathrm{sep}}]$. Suppose that $y \in L$ is not separable over $K(x_1, \ldots, x_n)$. Choose a minimal polynomial $F(z)$ for y over $K(x_1, \ldots, x_n)$. Then every exponent on z is divisible by p. Put each coefficient in lowest terms, and multiply $F(z)$ by a least common multiple of the denominators of the coefficients. This yields a polynomial $H(x_1, \ldots, x_n, z) \in K[x_1, \ldots, x_n][z]$ such that the coefficients in $K[x_1, \ldots, x_n]$ are relatively prime, and such that the polynomial is irreducible over $K(x_1, \ldots, x_n)[z]$. By Gauss's Lemma, this polynomial is irreducible in $K[x_1, \ldots, x_n, z]$. It cannot be the case that every exponent on every x_j is divisible by p, for if that were true, since the field is perfect, H would be a pth power, and not irreducible. By renumbering the x_i we may assume that x_n occurs with an exponent not divisible by p. Then the element x_n is separable algebraic over the field $K(x_1, \ldots, x_{n-1}, y)$, and we may use the transcendence basis $x_1, \ldots, x_{n-1}, y$ for L. Note that $x_n, y \in K(x_1, \ldots, x_{n-1}, y)^{\mathrm{sep}} = L''$, which is therefore strictly larger than $L' = K(x_1, \ldots, x_n)^{\mathrm{sep}}$. Hence, $[L : L''] < [L : L']$, a contradiction. $\square$

Lemma. *Let R be a ring.*

(a) *Let S be a direct limit of R-algebras S_j. Then $\Omega_{S/R}$ may be viewed as the direct limit of the modules $\Omega_{S_j/R}$ (or of the modules $S \otimes_{S_j} \Omega_{S_j/R}$).*

(b) *Let S be an R-algebra and let $T = S[X_i : i]$, a polynomial ring. Then*

$$\Omega_T/R \cong T \otimes_S \Omega_{S/R} \bigoplus \Big(\bigoplus_i T dX_i\Big),$$

and the value of $d_{T/R}$ on $F = \sum_{\mu \in \mathcal{M}} s_\mu \mu$, where μ runs through some finite set of monomials $\mathcal{M}$ in the X_i, is

$$\sum_{\mu \in \mathcal{M}} \mu \otimes d_{S/R}(s_\mu) + \sum_i \frac{\partial F}{\partial x_i} dX_i.$$

Hence, if $U = T/(F_j : j)$ then

$$\Omega_{U/R} \cong U \otimes_T \Omega_{T/R}/\langle d_{T/R}(F_j) : j \rangle,$$

where the brackets $\langle\ \rangle$ indicate span over T.

Proof. (a) One may deduce this using universal mapping properties. Here is another argument. Since S is the union of the images of the S_j, the images of the ds_j, $s_j \in S_j$, span $\Omega_{S/R}$. Each relation coming from addition, multiplication by a scalar in R, or the product rule in some S_j continues to hold when we map to $\Omega_{S/R}$, while it is clear that each such relation that holds in $\Omega_{S/R}$ comes from one that holds in some S_j.

In part (b), the second statement is immediate from the first. The formula for $d_{T/R}$ is forced by linearity and the product rule, and it is straightforward to verify that $d_{T/R}$ as defined is a derivation. (Note that in checking the product rule, one has that both sides are bilinear. Therefore it suffices to check it when each of the two elements is the product of an element of S with a monomial.) $\square$