Math 711: Lecture of September 29, 2004

Proposition. *Let $K \subseteq L$ be fields.*

(a) *If $L = K(x_i : i \in I)$ is a purely transcendental extension of K then $\Omega_{L/K}$ is the free L-module on the dx_i.*

(b) *If L' is a separable algebraic extension field of L, then $\Omega_{L'/K} \cong L' \otimes_L \Omega_{L/K}$.*

(c) *If $\{x_i : i \in I\}$ is a separating transcendence basis for L/K, then $\Omega_{L/K}$ is the free L-module on the basis dx_i.*

(d) *If K is perfect and L is finitely generated over K, then $\dim_L \Omega_{L/K}$ is the transcendence degree of L over K.*

Proof. Part (a) follows from the corresponding fact for polynomial rings together with the fact that localization commutes with formation of the module of differentials.

For part (b), a direct limit argument enables us to reduce to the case of a finite separable algebraic extension field L of K, and by the theorem on the primitive element, $L' = L[x]/(f)$ where f is separable over L. Let $f = \sum_t \lambda_t x^t$. By part (b) of the final Lemma from the Lecture Notes of September 27,

$$\Omega_{L'/K} \cong \big(L' \otimes_L \Omega_{L/K} \oplus L'dx\big)/L'd_{L[x]/K}f.$$

But

$$d_{L[x]/K}f = \sum_t d_{L/K}(\lambda_t)x^t + f'dx.$$

Since the image of f' in L is invertible, the quotient is simply the isomorphic with the module $L' \otimes_L \Omega_{L/K}$.

(c) is immediate from parts (a) and (b), while (d) is immediate from (c) and MacLane's theorem on the existence of separating transcendence bases. $\square$

We are now ready prove the characterization of étale extensions of fields stated in the Lecture Notes of September 27.

Proof of the theorem characterizing étale extensions of fields. We will prove that (c) $\Rightarrow$ (a) $\Rightarrow$ (b) $\Rightarrow$ (d) $\Rightarrow$ (c). Note that since R is finitely generated over a field, it and its quotients are finitely presented. For these rings, étale is equivalent to formally étale and unramified is equivalent to formally unramified.

The fact that (c) $\Rightarrow$ (a) follows from the Corollary to the first Proposition of the Lecture Notes of September 27 together with the Proposition from those same notes on behavior of finite products, while (a) $\Rightarrow$ (b) is immediate. To prove that (b) $\Rightarrow$ (d), note that $L \otimes_K R$ is unramified over L, and so it will suffice to prove that a finitely generated algebra over L is unramified if and only if it is isomorphic with a finite product of copies of L. We first claim that R must be zero-dimensional. If not, we may kill a minimal prime that is contained in a maximal ideal that is different from it, and so obtain an unramified finitely generated L-algebra that is a domain. The module of differentials of the fraction field

$\mathcal{F}$ has vector space dimension equal to the transcendence degree of $\mathcal{F}$ over L, which is the Krull dimension of R. But this must be zero. Therefore $\mathcal{F}$ must be equal to L, a contradiction.

Since R is zero-dimensional, it is a finite product of Artin local rings, and these may be considered separately. To complete the argument, we show that if an Artin local ring is a finitely generated L-algebra and is unramified, then it is $\cong L$. The residue field must be the image of L. If the maximal ideal is nonzero, we may kill $m^2 \neq m$ to obtain an example in which $m^2 = 0$ but $m \neq 0$. We may now kill all but one element in a minimal set of generators of m. The resulting ring has the form $R = L[x]/x^2$. But then $\Omega_{R/L}$ is the cokernel over R of the matrix $\bigl(2x\bigr)$, and is not zero since the image of $2x$ is in the maximal ideal of R, a contradiction. This concludes the proof that (b) $\Rightarrow$ (d).

It remains only to show that (d) => (c). Suppose that $L \otimes_K R$ is a finite product of copies of L. Then R is a finite-dimensional vector space over K. Since it is module-finite over K, it has Krull dimension zero. Since $R \subseteq L \otimes_K R$ (since L is free over K, this extension is faithfully flat), we see that R is reduced. Then R is a finite product of reduced Artinian local rings, each of which stays reduced when we apply $L \otimes_K \_$. It follows that each local ring of R is a finite algebraic field extension K' of K such that $L \otimes_K K'$ is reduced. It remains to show that K' is separable. Let $\theta \in K'$ have minimal polynomial $f = f(x)$ over K. Then $L \otimes_K K[\theta] \subseteq L \otimes_K K'$ is reduced, and $L \otimes_K K[\theta] \cong L \otimes_K K[x]/(f) \cong L[x]/(f)$ is reduced, which implies that f is square-free in $L[x]$, as required. $\square$

The following is a variant of Zariski's Main Theorem, and we shall refer to it as Zariski's Main Theorem.

Theorem (Zariski's Main Theorem). *Suppose that $R \subseteq S$ and that S is finitely generated as an R-algebra. Let Q be a prime ideal of S that is isolated in its fiber over P, a prime in R. Then there exists a module-finite extension R'' of R with $R \subseteq R'' \subseteq S$ and $f \in R'' - Q$ such that $R''_f = S_f$.*

Proof. Let R' be the integral closure of R in S and $P' = Q \cap R'$. Then Q is isolated in its fiber over P', and by the earlier version of ZMT, there exists $f \in R' - P' = R' - Q$ such that $R'_f = S_f$. Hence, for each of the finitely many generators u_j of S over R, we can choose N_j such that $f^{N_j}u_j = v_j/1$ with $v_j \in R'$. Let R'' be the subring of R' generated by f and the v_j. Clearly, $R'_f = S_f$. $\square$

Let $\mathcal{P}$ be a property of ring homomorphisms, and let $R \to S$ be a ring homomorphism. Let Q be a prime ideal of S lying over a prime P in R. We shall say that $\mathcal{P}$ *holds near* Q or that $R \to S$ *has* $\mathcal{P}$ *near* Q if there exist $b \in S - Q$ and $a \in R - P$ such that the image of a in S_b is invertible (so that there is an induced map R-algebra map $R_a \to S_b$: we say that $R_a \to S_b$ *is defined* in this case) and such that $R_a \to S_b$ has property $\mathcal{P}$. Thus, we may talk about a homomorphism $R \to S$ that is étale near Q, or smooth near Q, or unramified near Q, or formally unramified near Q, and so forth.

The following result makes major inroads in classifying étale and unramified morphisms.

Theorem. *Let S be an R-algebra, and Q a prime ideal of S lying over P in R.*

If S is finitely presented over R then $R \to S$ is unramified near Q if and only if there exist $a \in R - P$ and $b \in S - Q$ such that $R_a \to S_b$ is defined and S_b is a homomorphic image by a finitely generated ideal of a standard étale algebra over R_a.

If S is finitely generated over R then S is formally unramified near Q if and only if there exist $a \in R - P$ and $b \in S - Q$ such that $R_a \to S_b$ is defined and S_b is a homomorphic image of a standard étale algebra over R_a.

The proof involves Zariski's Main Theorem, the theorem on the primitive element for separable field extensions, our understanding of unramified homomorphisms when the base ring is a field, Nakayama's lemma, and additional trickery.

Before beginning the proof, we want to make several remarks. If $R \to S$ is finitely presented so is $R \to S_b$ and, hence, $R_a \to S_b$ when it is defined. Likewise, if S is finitely generated over R, then S_b is finitely generated over R and, hence, over R_a when $R_a \to S_b$ is defined. Also note that if we have maps $R \to R' \to S$ such that R' and S are finitely presented over R, then S is finitely presented over R'. (Take finitely many generators and relations for R' over R. Include the images of these generators and relations in a finitely many generators and relations for S over R. The additional generators and relations needed give a finite presentation of S over R'.)

Note that we already know the "if" part of the theorem: a standard étale algebra is unramified, and a quotient by an ideal is formally unramified and unramified if the ideal is finitely generated. Thus, we need only be concerned with proving the "only if" part.

The result in the final paragraph may be paraphrased as follows: if S is finitely generated over R it is formally unramified near Q if and only if it is a homomorphic image of a standard étale algebra near Q. The result in the second paragraph may be paraphrased as follows: if S is finitely presented over R, it is unramified near Q if and only if it is locally a homomorphic image, by a finitely generated ideal, of a standard étale algebra.

The statement in the second paragraph is immediate from the statement in the final paragraph: S_b will be finitely presented over the standard étale algebra, and therefore, if it is a homomorphic image of it, the ideal must be finitely generated.

Thus, we need only prove the "only if" part of the statement in the final paragraph.