Math 711: Lecture of October 1, 2004

Proof of the "only if" statement in the final paragraph of the theorem classifying unramified homomorphisms. Note first that if T is another finitely generated R algebra with an element $c \in T$ is such that $T_c \cong S_b$ where $b \notin Q$, then we may study T instead of S: T_c will have a prime ideal $Q'T_c$ corresponding to QS_b, where Q' is a prime ideal of T not containing c, and any localization of S_b at one element not in QS_b corresponds to a localization of T_c at one element not in QT_c.

We next want to observe that we may replace R and S by R_P and $R_P \otimes_R S = S_P$: this is a base change, and the hypotheses still hold. If we know the case where R is quasilocal then we know that for suitable elements $\alpha \in R_P - PR_P$ and $\beta \in S_P - QS_P$, we have that $(S_P)_\beta$ is a homomorphic image of a standard étale algebra over $(R_P)_\alpha$, say of $\big((R_P)_\alpha[x]\big)_g/(f)$, where f is monic over $(R_P)_\alpha$, g has coefficients in $(R_P)_\alpha$, and f' is invertible in the quotient. If we localize at the denominators in representations for α, β, the coefficients of f and g, and elements needed to show the invertibility of f' in $\big((R_P)_\alpha[x]\big)_g/(f)$ we get a standard étale algebra over a ring of the form R_a, namely $C = R_a[x]_g/(f)$ that will map to $(S_Q)_\beta$, where β may be assumed to be in $S - Q$. This map becomes onto if we invert all elements of $R - P$. Thus, we can find a single $a' \in R - P$ such that the image of $C_{a'}$ contains all of the generators in some finite set of generators for S over R, then map $C_{a'} \to S_{a'\beta}$ will be surjective.

Henceforth, we assume, as we may, that $(R,\, P, K)$ is quasi-local. Let $\overline{R}$ denote the image of R in S. Consider the fiber $K \to K \otimes S = S'$. This map is also formally unramified near the prime Q' corresponding to Q, which means that after localizing at one element b' of $S' - Q'$, we have an unramified map $K \to S'_{b'}$, and so Spec$\,(S'_{b'})$ is finite. This means that Q is isolated in its fiber over P: it is obviously minimal, and for finitely generated algebras over a field, maximal ideals contract to maximal ideals, and so it is maximal as well. Moreover, the field extension $K \to S/Q$ is a finite separable algebraic extension. We now apply the form of Zariski's Main Theorem proved in the Lecture Notes of September 29 to conclude that we have a module-finite R-algebra T with $R \subseteq T \subseteq S$ and an element $b \in T - Q$ such that $T_b = S_b$. It follows that T is also formally unramified over R near $Q \cap T$, and it will suffice to prove the theorem with S and Q replaced by T and $Q \cap T$. We may therefore assume without loss of generality that S is module-finite over R, where (R, P, K) is quasi-local.

Now the fiber S/PS is zero-dimensional. One of the factors, call it L, is S/Q, a finite separable algebraic extension of K. Let us write $S/PS = L \times B$, where B is simply the product of the other Artin local rings (there are finitely many) in the factorization of S/PS. Note that Q/PS corresponds to the ideal generated by $(0,\, 1)$, which is $\{0\} \times B$, in $L \times B$.

Let $\overline{\theta}$ denote a non-zero primitive element for L over K, so that $L = K[\overline{\theta}]$ (the condition that $\overline{\theta} \neq 0$ is automatic except in the case where $L = K$), and let $\theta \in S$ be an element that maps to $(\overline{\theta},\, 0)$ in $S/PS \cong L \times B$.

Let q be the contraction of Q to $\overline{R}[\theta]$. Note that $\theta \in S$ is integral over $\overline{R}$. We claim that $\overline{R}[\theta]_q = S_q = S_Q$ (once we know this, we will be able to replace S by $\overline{R}[\theta]$.) To prove that $\overline{R}[\theta]_q = S_q$ we may use Nakayama's lemma: since S is module-finite over R, S_q is module-finite over R_q. It therefore will suffice to show that $\overline{R}[\theta]_q/q\overline{R}[\theta]_q = S_q/qS_q$. We first consider what happens to the extension $\overline{R}[\theta] \subseteq S$ when we work mod the expansions of P, since $P\overline{R}[\theta] \subseteq q$. The image of $\overline{R}[\theta]/P\overline{R}[\theta]$ in S/PS is $\overline{R}[\theta]/(PS \cap \overline{R}[\theta]) \subseteq S/PS \cong L \times B$. Let $\theta' = (\theta, 0)$. Then the image of $\overline{R}[\theta]/(PS \cap \overline{R}[\theta])$ in $L \times B$ is $K[\theta']$, where we have identified K with its image in $L \times B$. The typical element of the image has the form $H(\theta') = (H(\theta), H(0))$, where H is an element of $K[x]$. Since we may choose H to be the minimal polynomial of $\theta \neq 0$, which has nonzero constant term α, the image contains $(0, \alpha)$, and hence, the image contains $e = (1,1) - \alpha^{-1}(0, \alpha) = (1, 0) \in L \times B$. The element e lies outside the contraction of Q/PS. Therefore, $(S/PS)_q$ is a localization of $(L \times B)_e \cong L$. Therefore, $(S/PS)_q = L$. It is also clear that $\overline{R}[\theta]/q = L$, since $\overline{\theta}$ is a primitive element for L over K. We have verified that $\overline{R}[\theta]_q = S_q$, and so S_q is already local with residue class field L. It follows as well $S_q = S_Q$.

We may now replace S by $\overline{R}[\theta]$. Let h denote the dimension of the K-vector space $K \otimes_R S = S/PS$, which is now spanned over K by the powers of the image of θ, which we call θ_1. Then $1, \theta_1, \theta_1^2, \dots, \theta_1^{h-1}$ is a K-vector space basis for $K \otimes_R S$. By Nakayama's lemma, it follows that $1, \theta, \theta^2, \dots \theta^{h-1}$ span S over R, and that there is a monic polynomial f in $R[x]$ such that if $\overline{f}$ denotes the image of f in $\overline{R}[x]$, then $\overline{f}(\theta) = 0$. Let C denote the ring $R[x]/\big(f(x)\big)$: we have that C maps onto S and that $K \otimes_R C \to K \otimes_R S$ is an isomorphism. Let $\mathfrak{n}$ denote the contraction of Q to C. Note that $C_\mathfrak{n}/\mathfrak{n}C_n \cong K[\overline{\theta}] \cong S_Q/QS_Q = L$.

We then have that $K \otimes_R \Omega_{C/R} \to K \otimes_R \Omega_{S/R}$ is an isomorphism, since the former is isomorphic to $\Omega_{K \otimes_R C/K}$, the latter to $\Omega_{K \otimes_R S/K}$, and $K \otimes_R C \cong K \otimes_R S$. Now,

$$L \otimes_C \Omega_{C/R} \cong (L \otimes_C C/PC) \otimes_C \Omega_{C/R} \cong L \otimes_C (K \otimes_R \Omega_C/R) \cong L \otimes_{C/PC} (K \otimes_R \Omega_C/R)$$

and, by an entirely similar argument, $L \otimes_S \Omega_{S/R} \cong L \otimes_{S/PS} \otimes (K \otimes_R \Omega_{C/R})$. Since $S/PS \cong C/PC$, we have that $L \otimes_C \Omega_{C/R} \cong L \otimes_S \Omega_{S/R}$, and so

$$(C_\mathfrak{n}/\mathfrak{n}C_\mathfrak{n}) \otimes_C \Omega_{C/R} \cong (S_Q/QS_Q) \otimes_S \Omega_{S/R} \cong (\Omega_{S/R})_Q/Q(\Omega_{S/R})_Q.$$

Since S_b is formally unramified over R, the numerator is 0, for even $\Omega_{S_b/R} \cong (\Omega_{S/R})_b = 0$. Thus,

$$0 = (C_\mathfrak{n}/\mathfrak{n}C_\mathfrak{n}) \otimes_C \Omega_{C/R} = L \otimes \Omega_{C_\mathfrak{n}/R},$$

and since $\Omega_{C_\mathfrak{n}/R}$ is finitely generated over $C_\mathfrak{n}$, we may apply Nakayama's lemma to conclude that $(\Omega_{C/R})_\mathfrak{n} \cong \Omega_{C_\mathfrak{n}/R} = 0$. Since $\Omega_{C/R}$ is finitely generated over C, we may choose a single element $g = g(x)$ with image not in $\mathfrak{n}$ such that $\Omega_{C_g/R} = 0$. Thus, C_g is unramified over R, and we may replace g by a multiple that maps to a multiple of b. We replace b by this multiple, and then we have a surjection $C_g \twoheadrightarrow S_b$. Since $C_g = R[x]_g/(f)$ is formally unramified over R and $\Omega_{C_g/R} = 0$ is the cokernel over C_g of the matrix $\big(f'\big)$ (one takes the image of f' in C_g), we have that the image of f' in invertible in $C_g = R[x]_g/(f)$, so that $R[x]_g/(f)$ is the required standard étale algebra. $\square$