Math 711: Lecture of October 4, 2004

We are now ready classify étale homomorphisms. First note that the property of $R \to S$ being flat is local on S: if S_Q is flat for every prime Q of S, so is S. For if $0 \to N \to M$ is an injection of R-modules, and $S \otimes_R N \to S \otimes_R M$ is not injective, the kernel is supported at some prime ideal Q of S, and $S_Q \otimes_R N \to S_Q \otimes_R M$ will have non-trivial kernel, a contradiction. Thus, if S is flat near Q for all prime ideals Q if and only if S is flat.

Theorem. *Let S be a finitely presented R-algebra. The following are equivalent:*
(a) *S is étale over R.*
(b) *S is flat over R and $\Omega_{S/R} = 0$.*
(c) *S is flat and unramified over R.*
(d) *Near every prime Q of $\mathrm{Spec}\,(S)$, S is standard étale over R (that is, there exists $b \in S - Q$ and $a \in R$ not in the contraction P of Q such that $R_a \to S_b$ is defined and is standard étale).*

Proof. We already know that (b) and (c) are equivalent. We next observe that (d) $\Rightarrow$ (a). Suppose that we have an R-algebra map $\phi : S \to T/J$ and we want to show that it has a unique lifting $S \to T$. This is very easy if we make use of the local nature of a map of schemes. For every prime Q of $\mathrm{Spec}\,(S)$ we can choose $b \in S - Q$ and $a \in R - P$, where P is the contraction of Q, such that $R_a \to S_b$ is defined and standard étale. Fix an element $\beta \in T$ that maps to the image of b in T/J. Then we have a surjection $T_\beta \twoheadrightarrow T_\beta/JT_\beta \cong (T/J)_b$. Note that T_β is independent of the choice of β: if β' also maps to the image of b, the images of the two differ by a nilpotent in T_β, and so β' also is invertible in T_β. The sets $D(b)$ cover $\mathrm{Spec}\,(S)$ (there is a choice of b outside any given prime Q), and for each of them we have a unique map $S_b \to T_\beta$ lifting the map $S_b \to (T/J)_b$. If we think in terms of schemes we have a map from the open set $\mathrm{Spec}\,(T_\beta) \to \mathrm{Spec}\,(S_b)$. These maps agree on overlaps: given b_1, b_2 and corresponding elements β_1, β_2, we may localize the map $S_{b_1} \to T_{\beta_1}$ at b_2 and the map $S_{b_2} \to T_{\beta_2}$ at b_1: the results must agree, because we get two liftings of $S_{b_1b_2} \to (T/J)_{b_1b_2}$ to $T_{\beta_1\beta_2}$, and $R \to S_{b_1b_2}$ is étale. Therefore there is a unique homomorphism $S \to T$ such that for all b, $S_b \to T_b$ is the same as $S_b \to T_\beta$. It is easy to check that this unique homomorphism gives the unique lifting.

The condition in (d) also implies that S is flat over R, since standard étale algebras are flat. Thus, (d) implies not only (a) but (b) and (c) as well.

To complete the proof, we shall show both that (a) $\Rightarrow$ (d) and that (c) $\Rightarrow$ (d). In either case, we have that S is unramified over R, and so after passing to a suitable choice of $R_a \to S_b$ (our hypothesis is preserved) we may assume that $S = C/I$ where $C = R[x]_g/(f)$ where f is monic and f' is invertible in C, i.e., C is standard étale over R. Let q be the contraction of Q to C. To complete the proof, it will suffice to show that I becomes 0 after localizing at some element of $C - q$. From the finite presentation condition, I is finitely generated. Therefore, it suffices to show that $I_q = 0$. We may now replace R by R_P and assume that $(R,\, P,\, K)$ is quasi-local.

By Nakayama's lemma, to show that $I_q = 0$ it will suffice to show instead that $I_q = I_q^2$, i.e., that $I_q/I_q^2 = 0$. Let $L = C_q/qC_q$. By another application of Nakayama's lemma it will suffice to show that $L \otimes_{C_q} (IC_q/I^2C_q) = 0$. Now, because C is unramified over R, C/PC is unramified over $R/P = K$, and so is reduced and zero-dimensional. But then C_q/PC_q is local, reduced, and zero-dimensional, which means that it is a field. It follows that $C_q/PC_q = L$ and that $PC_q = qC_q$. For any C_q-module M,

$$L \otimes_{C_q} M \cong (C_q/qC_q) \otimes_{C_q} M \cong M/qM \cong M/PM \cong R/P \otimes_R M = K \otimes_R M.$$

Applying this with $M = IC_q/I^2C_Q$, we see that to complete the proof it suffices to show that $K \otimes_R (IC_q/I^2C_q) = 0$.

We have an exact sequence of R-modules:

$$(**) \qquad 0 \to IC_q/I^2C_q \xrightarrow{\alpha} C_q/I^2C_q \xrightarrow{\beta} S_Q \to 0$$

(recall that $S_Q = C_q/IC_q$). If we are assuming (a) we have that S is étale over R and so S_Q is formally étale over R. Therefore the R-algebra isomorphism $S_Q \to C_q/IC_q$ has a unique lifting to a homomorphism $S_Q \to C_q/I^2C_q$. This map is a splitting, over R, of the map $\beta : C_q/I^2C_q \to S_Q$, so that $C_q/I^2C_q \cong S_Q \oplus IC_q/I^2C_q$. This implies that the exact sequence $(**)$ remains exact when we apply $K \otimes_R \_$. On the other hand, if we are assuming that S is flat over R then so is S_Q. This implies that $\mathrm{Tor}_R^1(K, S_Q) = 0$, and so the sequence $(**)$ remains exact when we apply $K \otimes_R \_$ as well. Thus, in either of the two cases, we see that we may identify $K \otimes_R (IC_q/I^2C_q)$ with the kernel of the map $K \otimes_R \beta$, and so we have reduced to proving that $K \otimes_R \beta$ is an injective map.

Since both $R \to C$ and $R \to C \twoheadrightarrow S$ are unramified over R, these two homomorphisms are unramified over K once we apply $K \otimes_R \_$. This implies that

$$K \to K \otimes_R C_q \qquad \text{and} \qquad K \to K \otimes_R C_q \twoheadrightarrow K \otimes_R S_Q$$

are maps of K into separable algebraic finite field extensions of K. We therefore have that the map $K \otimes_R C_q \to K \otimes_R S_Q$ is injective. Since we have a factorization

$$K \otimes_R C_q \twoheadrightarrow K \otimes_R C_q/I^2C_q \twoheadrightarrow K \otimes_R S_Q,$$

both these maps are injective as well as surjective, and the second of them is $K \otimes_R \beta$. $\square$