Math 711: Lecture of October 6, 2004

Our next main objective is the Jacobian criterion for smoothness. We need some preliminary results. The first gives a useful criterion for when the cokernel of a matrix over a quasilocal ring is free. For this lemma we need some discussion of ideals of minors.

If M is a matrix over a ring a ring R we denote by $I_t(M)$ the ideal generated by the size t minors (or subdeterminants) of M. By convention, $I_0(M) = R$. (The determinant of a 0×0 matrix ought to be 1, because it is the determinant of an identity map, albeit on a zero module. With this convention, the determinant of the direct sum of two square matrices is the product of their determinants, even if one of the matrices is 0×0.) Note that $I_t(M) = I_1\bigl(\bigwedge^t(M)\bigr)$. If M and N are matrices whose sizes are such that MN is defined, we have that $\bigwedge^t(MN) = \bigwedge^t(M)\bigwedge^t(N)$. It follows easily that every size t minor of MN is a sum in which each term is the product of a size t minor of M and one of N, and so $I_t(MN) \subseteq I_t(M)I_t(N)$. If U is invertible, $I_t(UM) \subseteq I_t(U)I_t(M) \subseteq I_t(M)$, while

$$I_t(M) = I_t\bigl(U^{-1}(UM)\bigr) \subseteq I_t(U^{-1})I_t(UM) \subseteq I_t(UM),$$

and so $I_t(M) = I_t(UM)$, provided, of course, that UM is defined. Similarly, if V is invertible and MV is defined we have that $I_t(M) = I_t(MV)$.

Lemma. *Let M be an $n \times m$ matrix over a quasilocal ring (A, q, K). Then the cokernel of the linear map induced by M from $A^m \to A^n$ (we also use M to denote this map) is free of rank d if and only if $I_{n-d+1}(M) = 0$ while $I_{n-d}(M) = A$.*

Moreover, if $d = n - m$ the following conditions are equivalent:

(a) Coker (M) *is free of rank $n - m$.*
(b) $I_m(M) = A$.
(c) *Some size m minor of A is a unit.*
(d) *The image of M mod q has rank m.*
(e) *The rows of M span A^m.*
(f) *There exists an $m \times n$ matrix $\mathcal{W}$ such that $\mathcal{W}M$ is the size m identity matrix.*

Proof. Replacing M by UMV, where U and V are invertible, does not affect the ideals of minors of various sizes, and does not affect the cokernel. If some entry of M is a unit we may perform elementary row and column operations until the unit is in the upper left hand corner and is replaced by the element 1. We may then perform elementary row and column operations to get the rest of the entries in the first column and row to be 0. We may then iterate this process with the $n - 1 \times m - 1$ matrix in the lower right corner. Eventually, we express M is the direct sum of a size k identity matrix $\mathbf{1}_k$ (k may be 0) and an $n - k \times m - k$ matrix M' all of whose entries are in q, and we need only consider this case. The cokernel of the M is the direct sum of the two cokernels, and therefore is equal to the cokernel of M'. This cokernel is free if and only if $M' = 0$, since otherwise one has non-trivial relations on what must be a minimal set of generators. Note that if $M' = 0$, the cokernel is free of rank $n - k$, and one has that $I_{k+1}(M) = 0$ while $I_k(M) = A$. Here,

$d = n - k$, and so $k = n - d$. It remains only to see that if $M' \neq 0$, then we cannot have one ideal of minors be the unit ideal while the ideal generated by the next larger size minors is 0. Consider the ideals of minors mod q. It is clear that, $I_k(M) = A$ while $I_t(M) \subseteq q$ for $t > k$. We can complete the proof of the first statement by showing that if $M' \neq 0$, then $I_{k+1}(M) \neq 0$. Let w be a nonzero entry of w and form the submatrix determined by the first k rows of M and the row of w together with the first k columns of M and the column of w. This submatrix is the direct sum of I_k and the 1×1 matrix $\bigl(w\bigr)$, and has determinant $w \neq 0$, as required.

The equivalence of the six conditions for the case $d = n - m$ is then easy: (a) $\Leftrightarrow$ (b) by what we have already proved, and (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) is clear. We have that (d) $\Rightarrow$ (e) by Nakayama's lemma, while (e) $\Rightarrow$ (f) is an easy exercise: the ith row of $\mathcal{W}$ consists of the coefficients needed to express the ith standard basis vector as a linear combination of the rows of M. Finally, (f) $\Rightarrow$ (b) because $A = I_m(\mathbf{1}_m) = I_m(\mathcal{W}M) \subseteq I_m(\mathcal{W})I_m(M) \subseteq I_m(M)$. $\square$

We also need:

Lemma. *Let S be an R-algebra and let T be any formally smooth R-algebra that maps onto S (we know that a polynomial ring over R or a localization of a polynomial ring is formally smooth). Let $S = T/I$. Then S is formally smooth over R if and only if the R-algebra homomorphism $T/I^2 \xrightarrow{\beta} S = T/I$ splits in the category of R-algebras.*

Proof. Evidently, if S is formally smooth one has a lifting of the identity on $S = T/I$ to an R-algebra map $\beta : S \to T/I^2$ which gives the splitting. Now suppose that one has this splitting and that one has a map $S \to U/J$ where $J^2 = 0$. Then we have a composite map $T \twoheadrightarrow S \to U/J$ from which we get an R-algebra map $T \to U$, lifting $T \to U/J$ (because T is formally smooth over R) such that I maps into J. Thus, I^2 maps into $J^2 = 0$, and so we have an induced map $T/I^2 \to U$ that lifts $T/I \to U/J$. The composite $S \xrightarrow{\beta} T/I^2 \to U$ gives the lifting we want. $\square$

We are now ready to prove the following:

Theorem (Jacobian criterion for smoothness). *Let $S = R[x]/I = R[x_1, \ldots, x_n]/I$, where I is finitely generated, and let $Q \in Spec(S)$. Let $\widetilde{Q}$ denote the inverse image of Q in $R[x]$. Let $h(Q)$ denote the least number of generators of $I_{\widetilde{Q}}$. Then $R \to S$ is smooth near Q if and only if $(\Omega_{S/R})_Q$ is free of rank $n - h(Q)$, and this holds iff S_Q is formally smooth over R.*

Moreover, a localization S' of S is formally smooth over R if and only if $(S')_Q$ is formally smooth for every prime (respectively, maximal) ideal Q of S'.

Proof. Consider a localization $S' = W^{-1}R[x]/I$, where $I = (F_1, \ldots, F_m)R[x]$. By the preceding Lemma, S' is formally smooth if and only if $W^{-1}R[x]/I^2 \to S$ splits. Let $\overline{x}_i$ be the image of x_i mod I^2 and x_i' be its image mod I. Constructing the splitting is equivalent to finding elements $\delta_1, \ldots, \delta_n \in I/I^2$ such the n elements $\overline{x}_1 + \delta_1, \ldots, \overline{x}_n + \delta_n$ can serve as the images of the x_i' under the splitting, and the condition is simply that the m elements

$$F_j(\overline{x}_1 + \delta_1, \ldots, \overline{x}_i + \delta_i, \ldots, \overline{x}_n + \delta_n)$$

vanish in $W^{-1}R[x]/I^2W^{-1}R[x]$. Using Taylor's formula, this system may be written as

$$F_j + \sum_i \frac{\partial F_j}{\partial x_i}\delta_i \equiv 0 \bmod I^2W^{-1}R[x].$$

That is, the expression on the left vanishes when each F_j and each $\dfrac{\partial F_j}{\partial x_i}$ is replaced by its image in $W^{-1}R[x]/I^2W^{-1}R[x]$.

Let $v_1, \ldots, v_m$ be the images of $F_1, \ldots, F_m$ in I/I^2, and let $v = (v_1 \ \ldots \ v_m)$. Let $\mathcal{J}$ be the image of the matrix $(\dfrac{\partial F_j}{\partial x_i})$ in S'. Since each δ_i is to be a linear combination of the elements v_j, say $\delta_i = \sum_{k=1}^m v_k w_{ki}$, with the $w_{ki} \in S'$, we see that S' is formally smooth over R if and only if there exists an $m \times n$ matrix $\mathcal{W}$ with entries in S' such that $-v = v\mathcal{W}\mathcal{J}$.

Here, v and $\mathcal{J}$ are fixed and we seek the unknown entries of $\mathcal{W}$. The system is a system of linear equations over S' in unknowns w_{ki} that are allowed to be arbitrary elements of S'. The coefficients are in $IW^{-1}R[x]/I^2W^{-1}R[x]$. The problem of whether such a system has a solution is local on S', that is, it has a solution in S' if and only if it has a solution after localization at every prime ideal if and only if it has a solution after localization at every maximal ideal. The reason is that such a system has a solution if and only if a certain element of a certain module is in the span of certain other elements of the module. The final statement of the theorem is now clear.

In the remainder of the proof we may assume that $W = R[x] - \widetilde{Q}$. Moreover, in analyzing this case we may assume that the F_j have been chosen to be a minimal system of generators of $IR[x]_{\widetilde{Q}}$. Thus, we are in the situation where $m = h(Q)$. The statement that $(\Omega_{S/R})_Q$ is free of rank $n - m$ is equivalent to the statement that $I_m(\mathcal{J})$ is the unit ideal. By part (f) of the Lemma on freeness of cokernels, if $I_m(\mathcal{J})$ is the unit ideal then we can choose $\mathcal{W}$ such that $\mathcal{W}(-\mathcal{J}) = \mathbf{1}_m$. On the other hand, if $-v = v\mathcal{W}\mathcal{J}$ then after tensoring with the residue class field (I/I^2 becomes an m-dimensional vector space with the images of the v_j as a basis), we see that the image of $\mathcal{J}$ has rank m. $\square$