Math 711: Lecture of October 8, 2004

We next want to give some alternative characterizations of smooth algebras.

Theorem. *Let S be a finitely presented R-algebra. The following conditions are equivalent:*

(a) *S is smooth over R.*
(b) *Near every prime Q of S, S is étale over a polynomial ring in finitely many variables.*
(c) *Near every maximal ideal Q of R, S is étale over a polynomial ring.*

In consequence, if S is smooth over R, then S is flat over R.

Proof. Polynomial and étale extensions are flat, and flatness is local on $Spec(S)$, so that condition (b) implies flatness. Therefore it will suffice to show that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a). It is clear that (b) $\Rightarrow$ (c). Assume (c). To prove that S is smooth, it suffices to prove that S_Q is formally smooth for every Q. Since every S_Q is a localization of S_m for some maximal $m \supseteq Q$, it suffices to show that S_m is formally smooth when m is maximal. This is clear from the condition in (c), since étale homomorphisms and adjunction of finitely many indeterminates are both smooth, and localization is formally smooth.

This means we need only prove the most interesting of the implications, namely, that (a) $\Rightarrow$ (b). Suppose that $S = R[x_1, \ldots, x_n]/I$ is smooth over R, where I is finitely generated. Fix $Q \in \mathrm{Spec}\,(S)$, and let $\widetilde{Q}$ be the inverse image of Q in $R[x]$. Suppose that $IR[x]_{\widetilde{Q}}$ has m generators. Then we may choose $g \in R[x] - \widetilde{Q}$ such that these m generators $F_1, \ldots, F_m$ are in $R[x]_g$ and $IR[x]_g = (F_1, \ldots, F_m)R[x]_g$. Here, $m = h(Q)$ in the notation of the Jacobian criterion from the preceding lecture. Let $\mathcal{J}$ denote the image of the matrix $(\dfrac{\partial f_j}{\partial x_i})$ over S. The fact that S is smooth near Q together with the Jacobian criterion for smoothness imply that some $m \times m$ minor of $\mathcal{J}$ is not in $\widetilde{Q}$. Without affecting any relevant issues, we may replace S by its localization at the image of this minor, and therefore assume that the image of this minor is invertible in S. By renumbering, we may assume that this minor is formed from the last m rows of $\mathcal{J}$.

Let $d = n - m$, and let $z_i = x_{i+d}$, $1 \leq i \leq m$. Let $R' = R[x_1, \ldots, x_d]$, and let G_j denote F_j thought of as an element of $R'[z_1, \ldots, z_m]$. Then we may think of S (after localizing at one element) as a localization at one element of $R'[z_1, \ldots, z_m]/(G_1, \ldots, G_m)$ such that the image of $(\dfrac{\partial G_j}{\partial z_i})$ is invertible: the determinant Δ is the minor that we know is not in $\widetilde{Q}$. This shows that

$$R'[z_1, \ldots, z_m]_{g\Delta}/(G_1, \ldots, G_m)$$

is étale over the polynomial ring $R' = R[z_1, \ldots, z_d]$, by part (b) of the first Proposition in the Lecture Notes of September 27. But this ring is the localization of S at the image b of $g\Delta$ $\square$

Before proceeding further, we want to characterize smooth extensions first over algebraically closed fields and then over arbitrary fields.

Lemma. *Let K be an algebraically closed field and let S be a finitely generated K-algebra. Let Q be a maximal ideal of S. The following conditions are equivalent:*

(a) *S_Q is (formally) smooth over K.*
(b) *S_Q is a regular local ring.*
(c) *$(\Omega_{S/K})_Q$ is free of rank equal to the Krull dimension of S_Q.*

Proof. S is generated over K by elements of Q ($S = K + Q$) and so we can map $K[x_1, \dots, x_n] \twoheadrightarrow S$ so that all x_i map into Q. Let $\widetilde{Q} = (x_1, \dots, x_n)$. Then $S_Q \cong W^{-1}K[x_1, \dots, x_n]/(F_1, \dots, F_m)$ where $W = K[x_1, \dots, x_n] - \widetilde{Q}$. Moreover, we may assume that the F_i are a minimal set of generators for the ideal they generate in the ring $W^{-1}K[x_1, \dots, x_n]$. The fact that (a) and (b) are equivalent may now be deduced from the Jacobian criterion for smoothness. Think of K as S/Q, and let $\mathcal{J}$ be the Jacobian matrix. Then $K \otimes_S \mathcal{J}$ has rank m if and only if the linear forms occurring in $F_1, \dots, F_m$ are linearly independent: the coefficient of x_i in F_j is the same as the image of $(\dfrac{\partial F_j}{\partial x_i})$ mod Q. This is equivalent to the condition that the F_j are part of a minimal system of generators for $\widetilde{Q}K[x_1, \dots, x_n]_{\widetilde{Q}}$, which is tested mod $\widetilde{Q}^2$. But we know that a quotient of a regular ring is regular if and only if the ideal being killed is part of a minimal set of generators of the maximal ideal: see Problem **4.** of Problem Set #2 from Math 615. This shows that (a) $\Leftrightarrow$ (b). Moreover, when these equivalent conditions hold, we have also seen that $n - m = \dim(S_Q)$, so that the equivalent conditions (a) and (b) imply (c).

Now assume (c), so that $(\Omega_{S/K})_Q$ is free of rank equal to $\dim(S_Q)$. We must show S_Q is regular. To see this, renumber the F_j so that $F_1, \dots, F_h$ have linearly independent linear forms, where h is the dimension of the span of the linear forms of the F_j. By subtracting K-linear combinations of $F_1, \dots, F_h$ from the remaining F_j, we may assume that $F_{h+1}, \dots, F_m$ are in $\widetilde{Q}^2$. Let $T = W^{-1}K[x_1, \dots, x_n]/(F_1, \dots, F_h)$. Then T is a regular local ring of Krull dimension $n-h$. Since the cokernel of $\mathcal{J}$ is free once we localize, the rank of $\mathcal{J}$ once we tensor with S_Q may be computed modulo QS_Q, and so is h. It follows that $(\Omega_{S/K})_Q$ has rank $n - h$. But we are given that the rank is $\dim(S_Q)$. It follows that $\dim(S_Q) = n - h$. But S_Q is a homomorphic image of the regular local ring T, which has dimension $n - h$ and is a domain. If the kernel were nonzero, the dimension of the quotient would drop. It follows that $S_Q = T$, and so S_Q is regular. Therefore (c) $\Rightarrow$ (b). $\square$