Math 711: Lecture of October 11, 2004

We next want to characterize smooth algebras over an arbitrary field. In order to do so, we need to discuss geometrically regular K-algebras. We need a lemma first.

Lemma. *Let $S \to T$ be a ring homomorphism.*

(a) *If T is faithfully flat over S then for every prime ideal P of S there is a prime ideal Q of T lying over P: in fact, any minimal prime of PT has this property.*

(b) *If $S \to T$ is a faithfully flat homomorphism and T is a regular Noetherian ring then S is regular.*

(c) *If S is Noetherian and is a direct limit $\varinjlim_t S_t$ of regular Noetherian rings, then S is regular.*

(d) *If $(S, P) \to (T, Q)$ is flat local, where the rings are Noetherian, then* $\dim(T) = \dim(S) + \dim(S/PS)$.

(e) *If $(S, P) \to (T, Q)$ is flat local such that (S, P) and and T/PT (the closed fiber) are regular, then T is regular.*

(f) *if $S \to T$ is flat, T is Noetherian, S is regular, and all fibers of $S \to T$ are regular, then T is regular.*

(g) *If $K \subseteq S$ where K is a field and L is a field containing K such that $L \otimes_K S$ is regular, then S is regular.*

(h) *If $K \subseteq S$ where K is a field and S is regular, and if L is a finite separable algebraic extension of K, then $L \otimes_K S$ is regular.*

Proof. For part (a), since T is faithfully flat PT is a proper ideal of T and has a minimal prime Q. We want to show that Q lies over P. Suppose that Q lies over P'. Then $R_{P'} \to S_Q$ is faithfully flat, and so is $R_{P'}/PR_{P'} \to S_Q/PS_Q$. This map is therefore injective. Since Q is minimal over PS, every element of the maximal ideal of S_Q/PS_Q is nilpotent, and so every element of the maximal ideal of $P'R_{P'}$ is nilpotent mod $PR_{P'}$. Since $PR_{P'}$ is prime, this shows that $P' = P$.

For (b), let P be any any prime of S. By part (a), we can choose Q lying over P, and then $S_P \to T_Q$ is a faithfully flat local map of local rings. Since T_Q is regular, so is S_P, by the second corollary on the first page of the Math 615 Lecture Notes from February 18.

In (c), let Q be any prime ideal of S and let Q_t be its contraction to S_t for all t. Then S_Q is the direct limit of the rings $(S_t)_{Q_t}$. Thus, we may assume without loss of generality that all the rings and maps are local. We use induction on $\dim(S)$.

Note that S is clearly a domain, since all the S_t are and a direct limit of domains is a domain. The case of dimension 0 is clear, since then S must be a field. If S has positive dimension let x be an element of $Q - Q^2$. Then for sufficiently large t_0, we have $x_{t_0} \in S_{t_0}$, mapping to x. We may restrict to $t \geq t_0$ without changing the direct limit. Let x_t be the image of x_{t_0} in S_t. Then we must have that $x_t \in Q_t - Q_t^2$: if x_t were a unit, x would be a unit, while if x_t were in Q_t^2, x would be in Q^2. Then the rings S_t/x_tS_t are regular, and their direct limit is S/xS. By the induction hypothesis, S/xS is regular, and, therefore, S is.

We prove (d) by induction on $\dim(S)$. If $\dim(S) = 0$ then P is nilpotent. Therefore, $\dim(T) = \dim(T/PT)$, as required. If $\dim(S) > 1$ we first reduce to the case where S is reduced: let J be the ideal of nilpotents in S and we may study $S/J \to T/JT$ instead. In the reduced case there is a nonzerodivisor x in S. Then x is a nonzerodivisor in T, since we may apply $T \otimes_S \_$ to $0 \to S \xrightarrow{x} S$, and we may apply the induction hypothesis to $S/xS \to T/xT$. The closed fiber is unaltered, and so $\dim(T) - 1 = \dim(T/xT) = \dim(S/xS) + \dim(T/PT)$ (by the induction hypothesis) $= \dim(S) - 1 + \dim(T/PT)$, and the result follows.

To prove (e), let d be the dimension of S and let n be the dimension of T/PT, so that T has dimension $d + n$ by part (e). P has d generators, and Q/PS has n generators, and putting together the former with liftings of the latter to Q, we see that Q has $d + n$ generators. This shows that T is regular.

For part (f), let Q be a maximal ideal of T lying over P in S. Then $S_P \to T_Q$ is faithfully flat and local, and the closed fiber is the fiber of $S \to T$ over P localized at Q, and so is regular. Thus, T_Q is regular by part (e).

Part (g) is immediate from (b), since $L \otimes_K S$ is faithfully flat over S (even free: L is free over K).

For part (h), note that since $S \to L \otimes S$ is flat, it suffices to prove that the fibers are regular, and each has the form $L \otimes_K K'$, where $K' = S_P/PS_P$ for some prime P. Let F be an algebraically closed field containing K'. Then it suffices to prove that $L \otimes_K F$ is regular, by part (g). But $L \cong K[x]/(f)$ where f is a polynomial with distinct roots in F, and so $L \otimes_K F \cong F[x]/(f)$ and, by the Chinese Remainder theorem, this is simply a product of copies of F, and is therefore regular. $\square$

In part (f), note that the result holds if we only assume that the fibers of $S \to T$ are regular over prime ideals P of S lying under maximal ideals of T.

Proposition. *Let K be a field and let S be a Noetherian K-algebra. The following conditions are equivalent:*

(a) *For every finite purely inseparable algebraic extension L of K, the ring $L \otimes_K S$ is regular.*
(b) *For every finitely generated field extension L of K, the ring $L \otimes_K S$ is regular.*

Moreover, if S is finitely generated over K the following conditions are also equivalent to these:

(c) *For some perfect field extension L of K, $L \otimes_K S$ is regular.*
(d) *For some algebraically closed field L with $L \supseteq K$, $L \otimes_K S$ is regular.*
(e) *For every field $L \supseteq K$, $L \otimes_K S$ is regular.*

Before we give the proof, we note that conditions (c), (d), and (e) cannot be used in the general case because the ring $L \otimes_K S$ need not be Noetherian.

Proof. We first note that if $L \subseteq L'$ are fields and $L' \otimes_K S$ is regular, then $L \otimes_K S$ is regular by part (g) of the Lemma. Evidently, (b) $\Rightarrow$ (a). For the other direction, note that given L, we may consider a field L' generated by finitely many generators for L over K over a perfect closure K' of K. Then L' has a separating transcendence basis over K', and can be obtained as a finite separable algebraic extension of a finite purely transcendental extension of K'. After replacing K' by a smaller field K_1 gotten by adjoining finitely many of elements of K' to K, we obtain a field L_1 finitely generated over K_1 (and, hence over K) which contains L and is obtained from K in three steps: a finite purely inseparable algebraic extension, then a finite transcendental extension, and finally, a finite separable algebraic extension. The first step gives a regular ring by hypothesis, the second obviously does not disturb regularity (one has a localization of a polynomial ring in finitely many variables), and the third preserves regularity by part (h) of the Lemma. This shows that (a) $\Rightarrow$ (b).

Now, when S is finitely generated over K, note that (e) $\Rightarrow$ (d) $\Rightarrow$ (c) $\Rightarrow$ (a) is clear (the last because a perfect extension contains every finite purely inseparable algebraic extension, coupled with part (g) of the Lemma), and we have already shown that (a) $\Rightarrow$ (b). Finally (b) $\Rightarrow$ (e) because every field extension is a direct limit of finitely generated field extensions, and we may apply $\_ \otimes_K S$ and use part (c) of the Lemma. $\square$

We shall say that a Noetherian K-algebra is *geometrically regular* if the equivalent conditions (a) and (b) of the Proposition hold for S. If S is a finitely generated K-algebra this property is characterized by the equivalent conditions (a) through (e) of the proposition.